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Non-commutative standard model: model building

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Abstract. A non-commutative version of the usual electro-weak theory is constructed. We discuss how to overcome the two major problems: (1) although we can have non-commutative $U(n)$ (which we denote by $U_\star(n)$) gauge theory we cannot have non-commutative $SU(n)$ and (2) the charges in non-commutative QED are quantized to just $0, \pm 1$. We show how the latter problem with charge quantization, as well as with the gauge group, can be resolved by taking the $U_\star(3) \times U_\star(2) \times U_\star(1)$ gauge group and reducing the extra $U(1)$ factors in an appropriate way. Then we proceed with building the non-commutative version of the standard model by specifying the proper representations for the entire particle content of the theory, the gauge bosons, the fermions and Higgs. We also present the full action for the non-commutative standard model (NCSM). In addition, among several peculiar features of our model, we address the *inherent CP* violation and new neutrino interactions.

1 Introduction

Undoubtedly, the usual particle physics standard model is among the most successful physical theories and so far it has passed all the precision tests and is capable of explaining all the present data, or those phenomena and concepts which can be accommodated within its mathematical structure, such as quarks and neutrino mass and mixing. The only unobserved, or perhaps theoretically less elegant, part is the Higgs sector.

Although being experimentally so successful, perhaps its only weak point is the large number of theoretically undetermined parameters. Mainly motivated by this point, there has been a lot of work devoted to formulating theories beyond the standard model, through which one can find some relations between the parameters of the standard model and in this way reduce the number of free parameters. Among these very different attempts one can mention the grand unified theories (GUTs) and the minimal supersymmetric standard model (MSSM).

In this work we construct a model beyond the standard model from a completely different perspective, i.e. the standard model on a non-commutative space-time, the non-commutative standard model (NCSM). Non-commutative space-time can be presented by the so-called Moyal plane, with the coordinates and their conjugate momentum operators, $\hat{x}_\mu,\ \hat{p}_\nu$, satisfying

$$
\begin{aligned}
[\hat{x}_\mu, \hat{x}_\nu] &= \mathrm{i}\theta_{\mu\nu} \quad , \quad \theta_{\mu\nu} = -\theta_{\nu\mu} \ , \\
[\hat{x}_\mu, \hat{p}_\nu] &= \mathrm{i}\hbar\eta_{\mu\nu} \quad , \quad [\hat{p}_\mu, \hat{p}_\nu] = 0 \ .
\end{aligned} \tag{1.1}
$$

In the above, $\theta_{\mu\nu}$, the non-commutativity parameter (usually taken as a constant tensor), is of dimension of $(\text{length})^2$. As it is seen, the Lorentz symmetry is lost, but we expect to find the manifest Lorentz symmetry at low energies, $E^2\theta \ll 1$ (at least if we ignore the quantum corrections), where θ is the dimensionful scale of the $\theta_{\mu\nu}$ tensor. Then, one should define field theory on the non-commutative space-times, non-commutative field theory. To pass to non-commutative field theories, it is enough to replace the usual product of the fields in the (commutative) action, by the Moyal $\star$-product[1]:

$$
\begin{aligned}
(f \star g)(x) &= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}} f(x)g(y)\Big|_{x=y} \\
&= f(x)g(x) + \frac{\mathrm{i}}{2}\theta_{\mu\nu}\partial_\mu f \partial_\nu g + \mathcal{O}(\theta^2) \ .
\end{aligned} \tag{1.2}
$$

Introducing this $\star$-product into the actions has some non-trivial consequences both at the classical (tree) and quantum (loop) levels.

At the classical level, among these consequences we would like to mention the restrictions it imposes on the gauge theories: only the non-commutative $U(n)$ gauge theories have a simple non-commutative extension and we cannot even have non-commutative $SU(n)$ gauge theories. Furthermore, the representations for the $u_\star(n)$ algebra are restricted to those of $n \times n$ hermitian matrices

[1] We note that this recipe cannot be used for gauge theories other than $U_\star(n)$

[1]. Also, non-commutativity imposes severe restrictions on the fermions and their charges [1,2]. We shall discuss these points in more detail in the next section. The other interesting classical consequence of non-commutativity is the inherent C and CP violation in the non-commutative field theories [3].

As for the quantum level, we can mention the loop calculations and renormalizability discussions. During the past two years there have been a large number of articles on that subject (see, e.g., [4–11,13,14])[2]. From all these results here we mention only two.

(i) In general, the unitarity of non-commutative field theories is related to having a space-like non-commutativity, i.e. $\theta_{\mu\nu}\theta^{\mu\rho}$ as a matrix should be positive definite [16];
(ii) an intrinsic and general feature of the non-commutative field theories is the so-called IR/UV mixing [7]: although we can usually remove the UV divergences in the non-commutative version of the usual commutative renormalizable theories by adding proper counter-terms (and hence the theory is UV renormalizable), upon sending the UV cut-off to infinity we are left with some new IR divergences. There have been three proposals to resolve this IR divergence problem [7,8,17,18], among which are the non-commutative hard resummation [17], and/or introducing a new way of regularization [18]; we believe that, one way or the other, this problem can be removed.

In particular we would like to point out that the non-commutative gauge theories [6,14], the non-commutative version of real ϕ^4 theory [7,9,10] as well as the complex ϕ^4 theory [11] and the non-commutative version of QED (NCQED) [2,12] have been shown to be one-loop renormalizable.

There have also been many attempts to study the phenomenological consequences of non-commutative field theories (by taking the space-time to be a non-commutative Moyal plane)[3]. However, most of them are aimed at accommodating the extra non-commutative contributions within the error bars of the present data [19,26,27]. A rigorous and robust mathematical framework which is not suffering from the charge quantization problem [1,2] and the extra $U(1)$ factors (in the $U_\star(n)$ gauge theory compared to $SU(n)$) [13,14] has not yet been constructed. This is exactly what we would like to do in this paper. We will show how, by fixing the gauge group of the non-commutative standard model (NCSM) to $U_\star(3)\times U_\star(2)\times U_\star(1)$ and reducing the two extra $U(1)$ factors through the appropriate Higgs particles, the number of possible particles in each family (there are six: left-handed leptons, right-handed charged leptons, left-handed quarks, right-handed up quarks, right-handed down quarks and Higgs) is fixed, as well as their hyper-charges (and hence the electric charge). We would like to emphasize that the existence of the Higgs particle, in our model, is an unavoidable outcome. As a consequence, two extra massive gauge bosons and two extra massive scalar particles will appear.

In order to make a distinction between the two types of scalar fields which we have: the one(s) which we use for the *reduction* of the extra $U(1)$ symmetries and the usual standard model Higgs, which is used for breaking the electro-weak symmetry, we call the former one "*Higgsac*" and keep the "Higgs" for the usual Higgs doublet[4].

This paper is organized as follows. In Sect. 2, we review the problems and restrictions for constructing a non-commutative version of the standard model and discuss a mechanism to resolve these problems. In Sect. 3, in order to show how our procedure works in practice, we work out the details of the reduction of the extra $U(1)$ factor(s) and show how this also resolves the charge quantization problem, for the particular case of the non-commutative version of QCD + QED which can be denoted by NC($SU_c(3) \times U(1)$) gauge theory. In Sect. 4, which in a sense is the main part of the paper, we construct the NCSM. We start with the $U_\star(3)\times U_\star(2)\times U_\star(1)$ gauge theory and reduce the two extra $U(1)$ factors by introducing two extra Higgsac particles in proper representations. Then, we proceed with introducing matter fields and discuss in detail how the hyper-charges are fixed to those of the usual standard model.

In Sect. 5, we work out the details of the electro-weak symmetry breaking. In this way we define the photon, Z and $W^\pm$ fields. Then, in the fermionic part, we discuss the interaction terms for the fermions and compare them with the usual standard model as well as the corresponding Yukawa couplings and mass terms. In Sect. 6, among several new features of NCSM, we mention the neutrino dipole moment and the non-commutative correction to the weak-mixing angle, $\theta_{\rm W}$, or more precisely to the ρ parameter and $\frac{m_W^2}{m_Z^2}$ ratio. In this way we impose some upper bounds on the masses of two extra massive gauge boson as well as on the non-commutativity parameter. Finally in Sect. 7, we discuss some of the open questions. A more detailed analysis of the normal sub-groups of $U_\star(n)$ as well as the Higgsac symmetry reduction is gathered in the appendices.

2 The major problems in constructing NCSM and the proposal to resolve them

In this section we recapitulate the problems one encounters in building a non-commutative version of the standard model and present the way out of them. These problems

[2] For a string theory survey on non-commutative issues, see [15]

[3] Non-commutative geometry (in a general sense) has been previously used to build a theory beyond the standard model; see e.g., [28]. Recently within the Connes formulation, the unimodularity condition has been used to obtain the hyper-charges for the fermions [29]. However, these models are based on a very different approach than ours, where the fields evolve in almost commutative spaces (the space-time is commutative with a minimal non-commutativity in the internal space)

[4] The suffix "ac" stems either from the word "acommutative" (i.e. not commutative) or from the diminutive suffix in Persian, similar to "ino" in Italian, and hence "Higgsac" is equivalent to "small Higgs". We use this terminology to distinguish these scalars from the usual Higgs and also the higgsinos of MSSM

Table 1. LH is for left handed, RH for right handed

Particles	Electric charge	$SU(2)$ weak charge	Hyper-charge	Color charge
LH electron	-1	$-\frac{1}{2}$	-1	none
LH neutrino	0	$+\frac{1}{2}$	-1	none
RH electron	-1	0	-2	none
LH up quark	$+\frac{2}{3}$	$+\frac{1}{2}$	$+\frac{1}{3}$	has
LH down quark	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{3}$	has
RH up quark	$+\frac{2}{3}$	0	$+\frac{4}{3}$	has
RH down quark	$-\frac{1}{3}$	0	$-\frac{2}{3}$	has
Higgs	0	$-\frac{1}{2}$	$+1$	none

and restrictions, which we classify in three sets, are all imposed by the mathematical (group theoretical) structure of non-commutative gauge theories. However, first let us review some related information about the usual standard model. The usual standard model in the gauge bosons sector contains eight (massless) gluons, one (massless) photon and three (massive) weak gauge bosons. We have collected the information about the matter fields and their charges in Table 1.

Now we are ready to discuss the three major problems.

2.1 Problems

(i) Charge quantization problem

As was shown in [2], the charges for the matter fields coupled to the $U_\star(1)$ theory must be quantized to just $0, \pm1$, depending on the representation of the particles. This is due to the fact that in a sense the $U_\star(1)$ theory is a non-Abelian theory (for a more detailed discussion we refer to [2,1]). Now, we face the first and the most challenging obstacle: As we explicitly see from Table 1, not all the electric or hyper-charges of the particles fulfill this condition. So, not only are we not able to construct NCQED, but going to the electro-weak level (and considering the hyper-charges) makes the problem worse and we face a larger variety of non-quantized hyper-charges.

(ii) The extra gauge fields

According to non-commutative group theoretical arguments (e.g. see [1]), the $U_\star(1)$ sub-group of $U_\star(n)$ is not a normal sub-group and therefore mathematically it is not possible to define a non-commutative $SU(n)$ algebra (or group) by simple insertion of $\star$-products. However, even if we ignore this mathematical fact and drop the corresponding $U_\star(1)$ gauge field in the $U_\star(n)$ gauge theory action, the remaining theory is not renormalizable [13,14]. Consequently, as a direct generalization of the $SU_c(3) \times SU_L(2) \times U(1)$ gauge theory, one cannot avoid two extra $U(1)$ factors, i.e., two extra gauge fields appearing in NCSM.

(iii) The no-go theorem

In [1], based on group theoretical arguments, we have proved a no-go theorem stating that

(a) the local $u_\star(n)$ *algebra* only admits the irreducible $n\times n$ matrix representation. Hence the gauge fields are in $n \times n$ matrix form, while the matter fields *can only be* in fundamental, adjoint or singlet states[5];
(b) for any gauge group consisting of several simple-group factors, the matter fields can transform non-trivially under *at most two* non-commutative group factors. In other words, the matter fields cannot carry more than two non-commutative gauge group charges.

The restriction (a) is actually what we have already had in the usual standard model, i.e. all the gauge bosons as well as the matter fields are living in the representations which are also allowed in the non-commutative case. However, as for criterion (b), it is clear from Table 1 that the particles coupled to gluons, the quarks, carry *three* different charges, i.e. hyper-charge, weak $SU(2)$ charge and color charge.

Before explaining our procedure to resolve the above-mentioned problems, however, we would like to make a comment on the no-go theorem. The arguments of [1], and in particular part (b), are based on the invariance of the action under *finite* gauge transformations. In other words, to define the gauge transformation for the matter fields we have considered the *group* factors, while in principle it is also possible to define these gauge transformations only with the *algebra* (i.e. the infinitesimal gauge transformations), in which case one can relax the condition (b)[6]. For the usual Lie groups and algebras where the group elements are obtained through the simple exponentiation of the algebra elements, of course the infinitesimal and finite gauge transformations result in the same physics (at least for Yang–Mills theories). However, this is not always the case, a famous example being the Chern–Simons theories in which, although the theory is invariant under infinitesimal gauge transformations, the invariance under finite gauge transformations is not immediate. As a result, to

[5] Within the superfield approach similar arguments have been presented in [30]

[6] We would like to thank L. Bonora for a discussion of this point

have a well-defined quantum Chern–Simons theory, the level should be quantized, which in turn is an implication of finite gauge transformations. For the non-commutative groups when the gauge group involves more than one simple $U_\star(n)$ factor, the relation between the algebra and the corresponding group is not given by a simple star exponentiation [1]. We believe that it is the invariance under the *finite* gauge transformations which is indeed fundamental, and of course this also covers the infinitesimal gauge invariance.

2.2 The way out

To show the way out of the above mentioned problems we recall two facts:

(I) In the usual physical models, there is always a $U(1)$ factor together with the $SU(n)$ factors, i.e. $SU_c(3)\times U_q(1)$ for QCD + QED and $SU_c(3)\times SU_{\mathrm{L}}(2)\times U_Y(1)$ for the standard model.

(II) If we define the photon (or the hyper-photon) through a linear combination of two (or three) $U_\star(1)$ fields, although the charge for each $U_\star(1)$ factor is quantized restrictively to 0 and ±1, there is a chance to find a higher variety of charges (but still quantized). Furthermore, this shows a way out of the implications of part (b) of our no-go theorem.

In agreement with our hopes there is a standard and well-known procedure to implement the above two facts: the Higgs symmetry-breaking scenario. Hence our recipe is to start with $U_\star(3)\times U_\star(1)$ (or $U_\star(3)\times U_\star(2)\times U_\star(1)$) and *reduce* two (or three) $U(1)$ factors to *one* $U(1)$ factor, through one (or two) proper Higgsac field(s). We would like to emphasize that in order to reduce a symmetry through the Higgs mechanism it is necessary that the Higgs is in a non-singlet representation of that symmetry. Therefore, in our case, the Higgsac field(s) should be charged *only* under the $U(1)$ sub-group of $U_\star(3)$ (and $U_\star(2)$) as well as under the individual $U_\star(1)$. Indeed the $U_\star(n)$ group enjoys the property of having the needed $U(1)$ normal sub-group, which here is denoted by $U_n(1)$. (For the definition of the $U_n(1)$ sub-group see Appendix A.)

In Sects. 3 and 4, we will explicitly and in detail show how the above observation works and how tightly it fits into the existing matter content of the standard model.

3 The non-commutative QED + QCD

To build the non-commutative version of the $SU_c(3)\times U(1)$ gauge theory, first we need to introduce the gauge group which will be denoted by NC$(SU(3)\times U(1))$. In order to achieve this goal and clarify the notation we need to describe the structure of the group $U_\star(n)$ in more detail. The $U_\star(n)$ stands for the usual non-commutative version of $U(n)$ obtained by insertion of the $\star$-product between the $U(n)$ matrix valued functions. Consequently, all $U_\star(n)$ matrix elements are power series in θ. Taking this into account, the $U_\star(n)$ has two invariant (normal) sub-groups:

(1) The group NC$SU(n)$ obtained from a $\star$-product of $SU(n)$ matrix valued functions (which do contain a $U(1)$ part, however, at least linear in θ so that in the limit $\theta\to0$ it reduces to the usual gauge group $SU(n)$). Therefore one can define the *factor*-group $U_n(1)=U_\star(n)/\mathrm{NC}SU(n)$. Note that $U_n(1)$ is a commutative Abelian sub-group;

(2) the $U_\star^n(1)$ sub-group, obtained by the action of $U_\star(n)$ on its $U_\star(1)$ sub-group. This $U_\star(1)$ sub-group is generated by star exponentiation of the trace of $u_\star(n)$ algebra elements, i.e. $\exp_\star(\mathrm{iTr}\lambda)\mathbf{1}_{n\times n},\quad \lambda\in u_\star(n)$, where the trace is taken over the $n\times n$ matrices. More explicitly, if $h\in$ the $U_\star(1)$ sub-group and $g\in U_\star(n)$, then the elements of $U_\star^n(1)$ are of the form of $g\star h\star g^{-1}$. We stress that this $U_\star(1)$ is not an invariant sub-group whereas $U_\star^n(1)$ is; and we emphasize that it is the factor-group $U_n(1)$ which is used in our standard model construction, while the other invariant sub-group, $U_\star^n(1)$, is not used throughout this paper. Also note that both of the NC$SU(n)$ and $U_\star^n(1)$ sub-groups should be understood as a power series in θ. The details of the sub-group construction are given in Appendix A.

To obtain the NC(QED + QCD) we start with the $U_\star(3)\times U_\star(1)$ gauge theory, establish the particle content and the representations, give the gauge transformations and write the gauge-invariant action. Subsequently, by a properly chosen Higgsac boson, we reduce the two existing $U(1)$ factors to a single $U(1)$ gauge symmetry, or, more precisely, the gauge group is reduced to NC$(SU(3)\times U(1))$. The final $U(1)$ factor will be proven to correspond to the non-commutative version of QED. Finally, we shall address the new features and interactions of NC(QED+ QCD), like CP violation, new "multi-photon" interactions and photon–gluon interactions.

3.1 The field content of the model; fixing the conventions

In the following, we shall fix our notations and also point out the fact that the $\star$-product will be omitted everywhere from now on, and unless mentioned explicitly, it is understood that the $\star$-product is there.

The pure $U_\star(3)\times U_\star(1)$ theory is described by one gauge field, B_μ, valued in the $u_\star(1)$ algebra and the $u_\star(3)$-valued gauge fields:

$$G_\mu(x)=\sum_{A=0}^{8}G_\mu^A(x)T^A\ . \tag{3.1}$$

According to [1], the gauge fields corresponding to $u_\star(3)$ are necessarily in a 3×3 matrix form, because no other representation for the $u_\star(3)$ algebra is possible. As a result, we can take the generators $T^a,\ a=1,2,\cdots,8$ to be the Gell-Mann matrices, while $T^0=\mathbf{1}_{3\times3}$.

If we denote the elements of $U_\star(1)$ by $v(x)$ and the elements of $U_\star(3)$ by $U(x)$, we can write the finite local transformations of the gauge fields as

$$B_\mu\to vB_\mu v^{-1}+\frac{\mathrm{i}}{g_1}v\partial_\mu v^{-1}\ ,$$

$$G_\mu \to U G_\mu U^{-1} + \frac{\mathrm{i}}{g_3} U \partial_\mu U^{-1} \ . \tag{3.2}$$

Then the gauge field strengths

$$\begin{aligned} B_{\mu\nu} &= \partial_{[\mu} B_{\nu]} + \mathrm{i} g_1 [B_\mu, B_\nu]_\star \ , \\ G_{\mu\nu} &= \partial_{[\mu} G_{\nu]} + \mathrm{i} g_3 [G_\mu, G_\nu]_\star \end{aligned} \tag{3.3}$$

will transform as $B_{\mu\nu} \to v B_{\mu\nu} v^{-1}$ and $G_{\mu\nu} \to U G_{\mu\nu} U^{-1}$, leaving the action of the $U_\star(3)\times U_\star(1)$ Yang–Mills theory,

$$S_{\mathrm{NCYM}} = -\frac{1}{4} \int \mathrm{d}^4 x [B_{\mu\nu} B^{\mu\nu} + \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu})] \ , \tag{3.4}$$

invariant. A full account of these issues, along the lines of the scope of this paper, is given in [1,14].

As for the matter content of the $U_\star(3)\times U_\star(1)$ theory, the number of independent charged particles that can occur in this model, according to the no-go theorem [1], is $\frac{1}{2} 2 \times (2+1) = 3$, since the number of simple-group factors is two. We take these particles to be the electron (in the anti-fundamental representation of $U_\star(1)$), the up quark (in the fundamental representation of $U_\star(1)$ and the anti-fundamental representation of $U_\star(3)$) and the down quark (in the anti-fundamental representation of $U_\star(3)$). Then the gauge transformation properties of the fermions under the $U_\star(3)\times U_\star(1)$ gauge group are

$$\begin{aligned} \psi_e(x) &\to \psi_e(x) v^{-1}(x) \ , \\ \psi_u(x) &\to v(x) \psi_u(x) U^{-1}(x) \ , \\ \psi_d(x) &\to \psi_d(x) U^{-1}(x) \ . \end{aligned} \tag{3.5}$$

The gauge-invariant action corresponding to this $U_\star(3)$ $\times U_\star(1)$ model is

$$\begin{aligned} S = \int \mathrm{d}^4 x \bigg[& \bar{\psi}_e \gamma^\mu D^1_\mu \psi_e + \bar{\psi}_u \gamma^\mu D^{1+3}_\mu \psi_u + \bar{\psi}_d \gamma^\mu D^3_\mu \psi_d \\ & - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} \mathrm{Tr}(G^{\mu\nu} G_{\mu\nu}) \bigg] \ . \end{aligned} \tag{3.6}$$

The covariant derivatives entering (3.6) are

$$D^1_\mu = \partial_\mu - \frac{\mathrm{i}}{2} g_1 B_\mu \ , \tag{3.7}$$

$$D^3_\mu = \partial_\mu - \frac{\mathrm{i}}{2} g_3 G'^0_\mu - \frac{\mathrm{i}}{2} g_3 G^a_\mu T^a \ , \tag{3.8}$$

$$D^{1+3}_\mu = \partial_\mu + \frac{\mathrm{i}}{2} g_1 B_\mu - \frac{\mathrm{i}}{2} g_3 G'^0_\mu - \frac{\mathrm{i}}{2} g_3 G^a_\mu T^a \ . \tag{3.9}$$

For a reason that will become clear in a moment, we have denoted the zeroth component of the $U_\star(3)$ gauge field by G'^0_μ.

Still, this is not NC(QCD + QED), but a theory that suffers from the charge quantization problem. In order to cure it, we use the Higgs procedure for reducing the extra $U(1)$ factors of $U_\star(3)\times U_\star(1)$ to a single $U(1)$, which will exhibit the properties of a true non-commutative version of QED in the coupling of the non-commutative photon to the fermionic fields.

The reduction of symmetry has to be done through a proper Higgsac field, i.e. a scalar particle that is charged under those groups (or sub-groups) that we intend to reduce. In this case, the scalar field has to be charged under the $U_1(1)$ and $U_3(1)$ invariant sub-groups of the $U_\star(1)$ and $U_\star(3)$ factors. The gauge transformation undergone by the symmetry-breaking scalar, Higgsac field, is

$$\Phi(x) \to U_1(x) \Phi(x) v^{-1}(x) \ , \tag{3.10}$$

where $U_1(x)$ is the *θ-independent* phase factor of $U_3(1)$ and $v(x) \in U_1(1)$ (for more details see Appendix A). We should stress that $\Phi(x)$ is θ-independent and in (3.10) the usual product (and not the $\star$-product) should be used. Since the NC$SU(3)$ and hence $U_3(1)$ sub-groups should be understood as power series expansions in θ, the symmetry reduction problem should be investigated systematically in the same power series. We stress that the $U_1(1)$ and $U_3(1)$ phase factors are $U_\star(3)\times U_\star(1)$ invariant. The details of the symmetry reduction are given in Appendix B.

The only gauge-invariant terms introduced in the gauge-invariant action by the presence of the scalar field are

$$(D^{1+1}_\mu \Phi)^\dagger (D^{1+1}_\mu \Phi) + m^2 \Phi^\dagger \Phi - \frac{f}{4!} (\Phi^\dagger \Phi)^2 \ , \tag{3.11}$$

with the covariant derivative given by

$$D^{1+1}_\mu = \partial_\mu + \frac{\mathrm{i}}{2} 3 g_3 G'^0_\mu - \frac{\mathrm{i}}{2} g_1 B_\mu \ , \tag{3.12}$$

where by G'^0, B in the above we only mean the θ-independent parts of the corresponding gauge fields. These θ-independent parts are those which transform properly under $U_\star(3) \times U_\star(1)$. Note that in (3.11) and (3.12) the usual product of functions should be used.

Applying the usual Higgs mechanism, we shall obtain a massive gauge boson, G^0_μ, whose mass term in the Lagrangian is

$$\frac{1}{4} (3 g_3 G'^0_\mu - g_1 B_\mu)^2 \phi_0^2 = N^2 (G^0_\mu)^2 \phi_0^2, \tag{3.13}$$

where $N = \frac{1}{2} \sqrt{g_1^2 + (3 g_3)^2}$ is a normalization factor and ϕ_0 is the vacuum expectation value for the scalar field. Actually, in order to write (3.13), we have performed a rotation in the (B_μ, G'^0_μ) plane by the angle δ_{13},

$$\tan \delta_{13} = \frac{g_1}{3 g_3} \ , \tag{3.14}$$

so that

$$\begin{aligned} G^0_\mu &= \cos\delta_{13} G'^0_\mu - \sin\delta_{13} B_\mu, \\ A_\mu &= \sin\delta_{13} G'^0_\mu + \cos\delta_{13} B_\mu \ , \end{aligned} \tag{3.15}$$

where A_μ is the (massless) non-commutative photon, i.e. the gauge field of the residual $U(1)$ symmetry. The reciprocal of this rotation is given by

$$\begin{aligned} G'^0_\mu &= \cos\delta_{13} G^0_\mu + \sin\delta_{13} A_\mu \ , \\ B_\mu &= -\sin\delta_{13} G^0_\mu + \cos\delta_{13} A_\mu \ . \end{aligned} \tag{3.16}$$

As desired the Lagrangian (3.23) is $U_\star(3)\times U_\star(1)$ gauge invariant. However, the Higgsac field may interact with other matter fields *only* indirectly, via the θ-independent parts of the corresponding gauge fields. Here we only investigate these effects in the leading order. Therefore in this leading order the theory should be treated as an "effective theory" for energies lower than the non-commutativity scale, which, as we will discuss, can be as low as TeV. From this point of view our model is an effective theory up to the TeV scale. The calculation of higher θ-corrections would require a more detailed analysis which is postponed to future works.

3.2 Reduction of the $U_\star(1)$ symmetries: a solution to the charge quantization problem

Now we show that this Higgs mechanism has indeed brought us to the NC(QCD + QED), by curing the charge quantization problem that plagues the usual $U_\star(1)$ gauge theory. To this end, we show that the fermions of the $U_\star(3)\times U_\star(1)$ theory couple to the massless gauge boson of the residual $U_\star(1)$, A_μ, through the usual electric charges (see Table 1).

For the electron, the coupling to A_μ emerges from the first term of (3.6), taking into account (3.16),

$$\begin{aligned}\bar{\psi}_e\gamma^\mu D^1_\mu\psi_e &= \bar{\psi}_e\gamma^\mu\partial_\mu\psi_e - \frac{\mathrm{i}}{2}g_1\bar{\psi}_e\gamma^\mu\psi_e B_\mu \qquad (3.17)\\ &= \bar{\psi}_e\gamma^\mu\partial_\mu\psi_e - \frac{\mathrm{i}}{2}g_1\cos\delta_{13}\bar{\psi}_e\gamma^\mu\psi_e A_\mu + \cdots\ ,\end{aligned}$$

where the dots indicate the coupling to the massive gauge boson, G^0_μ. We would like to remind the reader that, although it is not shown explicitly, the products between the fields are all performed by the Moyal star product.

As we want the term relevant for the coupling of the electron to the gauge field A_μ to be proportional to the electric charge of the electron, i.e. $-e$, we define e as

$$\frac{1}{2}g_1\cos\delta_{13} = e\ . \qquad (3.18)$$

A similar reasoning for the down quark will give

$$\begin{aligned}\bar{\psi}_d\gamma^\mu D^3_\mu\psi_d &= \bar{\psi}_d\gamma^\mu\partial_\mu\psi_d - \frac{\mathrm{i}}{2}g_3\bar{\psi}_d\gamma^\mu\psi_d G'^0_\mu\\ &\quad - \frac{\mathrm{i}}{2}g_3\bar{\psi}_d\gamma^\mu\psi_d G^a_\mu T^a\\ &= \bar{\psi}_d\gamma^\mu\partial_\mu\psi_d - \frac{\mathrm{i}}{2}g_3\sin\delta_{13}\bar{\psi}_d\gamma^\mu\psi_d A_\mu\\ &\quad - \frac{\mathrm{i}}{2}g_3\bar{\psi}_d\gamma^\mu\psi_d G^a_\mu T^a + \cdots\ , \qquad (3.19)\end{aligned}$$

from which we find the condition

$$-\frac{1}{2}g_3\sin\delta_{13} = q_d\ , \qquad (3.20)$$

where q_d is the electric charge of the down quark. However, using (3.14) and (3.18) we find that

$$q_d = -\frac{1}{3}e\ , \qquad (3.21)$$

which is the correct relation.

For the up quark,

$$\begin{aligned}\bar{\psi}_u\gamma^\mu D^{1+3}_\mu\psi_u &= \bar{\psi}_u\gamma^\mu\partial_\mu\psi_u + \frac{\mathrm{i}}{2}g_1\bar{\psi}_u\gamma^\mu B_\mu\psi_u \qquad (3.22)\\ &\quad - \frac{\mathrm{i}}{2}g_3\bar{\psi}_u\gamma^\mu\psi_u G'^0_\mu - \frac{\mathrm{i}}{2}g_3\bar{\psi}_u\gamma^\mu\psi_u G^a_\mu T^a\end{aligned}$$

and the relevant terms for the coupling with A_μ, having in view (3.16), will be

$$\begin{aligned}\mathcal{L}_{u-A_\mu} &= \frac{\mathrm{i}}{2}g_1\cos\delta_{13}\bar{\psi}_u\gamma^\mu A_\mu\psi_u - \frac{\mathrm{i}}{2}g_3\sin\delta_{13}\bar{\psi}_u\gamma^\mu\psi_u A_\mu\\ &= \frac{\mathrm{i}}{2}(g_1\cos\delta_{13} - g_3\sin\delta_{13})\bar{\psi}_u\gamma^\mu A_\mu\psi_u\\ &\quad - \frac{\mathrm{i}}{2}g_3\sin\delta_{13}\bar{\psi}_u\gamma^\mu[\psi_u, A_\mu]_\star\ , \qquad (3.23)\end{aligned}$$

and therefore

$$\frac{1}{2}(g_1\cos\delta_{13} - g_3\sin\delta_{13}) = q_u\ . \qquad (3.24)$$

Upon using the definition of e in (3.18) and (3.14), we find

$$q_u = +\frac{2}{3}e\ . \qquad (3.25)$$

As we see, the charges for the up and down quarks have come out of the mathematical structure of our model and they have *not* been put by hand. In fact, the only allowed (possible) charges for the particles which also couple to the gluons are $\frac{1}{3}$ and $\frac{2}{3}$ in units of electron charge. In other words, the representation fixes completely the electric charges. The reader may find some more details on the symmetry reduction in the fermionic sector in Appendix C.

3.3 Discussions of the model; some new features

Although we do not analyze the NC($SU_c(3)\times U(1)$) model described previously in detail, we would like to point out some of the important consequences and a more detailed survey is postponed to future works.

(1) The renormalizability

Noting the fact that in order to construct our model we started with a $U_\star(3)\times U_\star(1)$ gauge theory plus all possible charged matter fields, this theory is (UV) renormalizable [2,14]. In addition we have used a (complex) scalar field coupled to the two commutative $U(1)$ factors with a $(\phi^\dagger\phi)^2$ potential and it is well known that this scalar theory is renormalizable. On the other hand, it is well known that the Higgs scenario does not spoil the renormalizability of the theory. Hence, altogether we expect our theory to be renormalizable.

(2) The photon–photon and photon–gluon interactions

Having the definition of physical fields, A_μ, G^0_μ, in terms of B_μ, G'^0_μ, we can easily read off the interaction of the photon with itself and also with other gauge bosons. Inserting (3.16) into the action (3.4), there are some immediate results.

(i) The three- and four-photon vertices are *not* exactly as dictated by a simple $U_\star(1)$ theory. The coefficient (coupling) for the A_μ–A_μ–A_μ term is $\frac{2}{3}e(1+2\sin^2\delta_{13})$, while for the A_μ–A_μ–A_μ–A_μ term it is $\frac{4}{3}e^2(1+2\sin^2\delta_{13})$;
(ii) there are A_μ–G^0_μ–G^0_μ, A_μ–A_μ–G^0_μ and A_μ–A_μ–G^0_μ–G^0_μ interaction terms;
(iii) the usual gluons (the G^a_μ, $a=1,\cdots,8$ fields) also couple to the photon, A_μ.

As a side effect of the above arguments it is likely that they show a way out of the standing problem of a simple $U_\star(1)$ gauge theory: the negative β-function [2]. It is an experimentally confirmed fact that the QED coupling, α, increases as we increase the energy:

$$\alpha|_{E\sim 10\,\mathrm{eV}} \simeq \frac{1}{137.036}\ ,\qquad \alpha|_{E\sim m_Z} \simeq \frac{1}{128.9}\ . \tag{3.26}$$

On the other hand, a direct one-loop calculation for the simple $U_\star(1)$ gauge theory shows a negative β-function. However, according to our arguments one should keep in mind that in the NC($SU_c(3)\times U(1)$) model discussed above, the photon is also involved in some interactions other than those of the $U_\star(1)$ theory. Also, the number of charged particles coupled to the photon is now increased, as the charge quantization problem of the quarks has been eliminated. This may show a way to resolve the negative β-function problem.

(3) The fermionic interactions

Here we would like only to mention the inherent CP violation because of the $\star$-product present in the fermion–photon coupling terms. As discussed in [3], it is important that the photon appears on the right-hand side (or left-hand side) of the ψ field, like the up quark (or electron and down quark). Consequently, the anti-particle of the up quark (which carries $-\frac{2}{3}e$ charge) would be coupled to a photon from the left-hand side. More intuitively, the non-commutative particles, besides the usual electric charges, also carry higher-pole (including dipole) moments [31,12]; the anti-particle of any particle not only should carry the opposite charge, but also the opposite dipole moment. Since these dipole moments are proportional to the momentum [31,12], the theory would not be CP invariant, while CPT is conserved [3].

Finally, we would like to note that in the up quark–photon interaction term (3.23), besides the usual $\bar{\psi}\gamma^\mu\psi A_\mu$ term, there is a Moyal bracket term which is not there for electron and down quark. Group theoretically, this is related to the fact that the up quark carries two different charges, while the electron and down quark carry only one type of charge.

4 The non-commutative standard model (NCSM)

Having worked out the details of the $U_\star(3)\times U_\star(1)$ gauge theory, the symmetry reduction scenario and the charges of the particles as a warm up, we are now ready to present our formulation of NCSM. In this section, applying the same machinery, but for the group $U_\star(3)\times U_\star(2)\times U_\star(1)$ we construct the NCSM. First we show the reduction of three $U(1)$ factors to the *hyper-charge* $U(1)$ and discuss that, as a result, two of the corresponding $U(1)$ fields become massive. Then, we proceed with the matter fields and show that their hyper-charges are fixed to those of the usual standard model (given in Table 1).

4.1 The gauge group

The pure $U_\star(3)\times U_\star(2)\times U_\star(1)$ theory is described by one gauge field, B_μ, valued in the $u_\star(1)$ algebra, the $u_\star(2)$-valued gauge fields:

$$W_\mu(x)=\sum_{I=0}^{3} W^I_\mu(x)\sigma^I\ , \tag{4.1}$$

and the $u_\star(3)$-valued gauge fields:

$$G_\mu(x)=\sum_{A=0}^{8} G^A_\mu(x)T^A\ . \tag{4.2}$$

For a similar reason as in the previous section, i.e. according to the no-go theorem [1], we take the generators of the $u_\star(2)$ algebra as the Pauli matrices σ^i, $i=1,2,3$ and $\sigma^0=\mathbf{1}_{2\times 2}$, while the generators of the $u_\star(3)$ algebra will be taken as the Gell-Mann matrices T^a, $a=1,2,\cdots,8$ and $T^0=\mathbf{1}_{3\times 3}$.

In the following we continue to denote the elements of $U_\star(1)$ by $v(x)$ and the elements of $U_\star(3)$ by $U(x)$, while the elements of $U_\star(2)$ are denoted by $V(x)$. The local transformations of the gauge fields are of a similar form as (3.2) and the action

$$S_{\text{gauge fields}} \tag{4.3}$$
$$=-\frac{1}{4}\int \mathrm{d}^4x[B_{\mu\nu}B^{\mu\nu}+\mathrm{Tr}(W_{\mu\nu}W^{\mu\nu})+\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})]$$

is gauge invariant.

In order to reduce the three $U(1)$ factors of the $U_\star(3)\times U_\star(2)\times U_\star(1)$ theory we should use two scalar particles and run the Higgs mechanism two times. One single Higgsac cannot do the task, because the scalar particle used for reducing a symmetry should be charged under the symmetry group we want to reduce. In our case, these symmetry groups are the $U(1)$ factor-groups of $U_\star(1)$, $U_\star(2)$ and $U_\star(3)$. Therefore, we begin by first reducing the $U(1)$ subgroups of $U_\star(2)$ and $U_\star(3)$ to some residual $U(1)$ whose corresponding (massless) gauge field will be denoted by B'_μ. Subsequently, this symmetry and the individual $U_\star(1)$

will be reduced to the $U(1)$ corresponding to the hypercharge, described by the gauge field Y_μ.

Let us start by choosing the first symmetry-reducing scalar particle with the transformation properties:

$$\Phi_1(x) \to U_1(x)\Phi_1(x)V_1^{-1}(x) \ , \tag{4.4}$$

where U_1 stands for the elements of the $U_3(1)$ sub-group of $U_\star(3)$ and V_1 stands for the elements of the $U_2(1)$ sub-group of $U_\star(2)$. We note that these sub-groups are constructed in the same way as in the previous section and in (4.4) the products are the usual commutative ones.

The covariant derivative corresponding to this scalar field is

$$D_\mu = \partial_\mu + \frac{\mathrm{i}}{2}3g_3G_\mu^{'0} - \frac{\mathrm{i}}{2}2g_2W_\mu^{'0} \ . \tag{4.5}$$

Note that here the covariant derivative only involves the θ-independent parts of the corresponding gauge fields. The Lagrangian for the Φ_1 field will acquire the new terms:

$$(D_\mu\Phi_1)^\dagger(D_\mu\Phi_1) + m_1^2\Phi_1^\dagger\Phi_1 - \frac{f_1}{4!}(\Phi_1^\dagger\Phi_1)^2 \tag{4.6}$$

(with no $\star$ product), which, as desired, is fully gauge invariant (for more details see Appendix A). Through the Higgs mechanism, we obtain a mass term for the gauge boson G_μ^0:

$$\left(\frac{3}{2}g_3G_\mu^{'0} - g_2W_\mu^{'0}\right)^2 \phi_1^2 = N_1^2(G_\mu^0)^2\phi_1^2, \tag{4.7}$$

where $N_1 = \sqrt{g_2^2 + \left(\frac{3}{2}g_3\right)^2}$ and $\phi_1 = \sqrt{\frac{12m_1^2}{f_1}}$ is the vacuum expectation value for the Φ_1 Higgsac field.

The massive gauge boson G_μ^0 and the residual massless $U_1(1)$ field, B'_μ, can be defined through a rotation by the angle δ_{23},

$$\tan\delta_{23} = \frac{2g_2}{3g_3} \tag{4.8}$$

in the $(W_\mu^{'0}, G_\mu^{'0})$ plane, i.e.

$$\begin{aligned} G_\mu^0 &= \cos\delta_{23}G_\mu^{'0} - \sin\delta_{23}W_\mu^{'0}, \\ B'_\mu &= \sin\delta_{23}G_\mu^{'0} + \cos\delta_{23}W_\mu^{'0} \ , \end{aligned} \tag{4.9}$$

whose reciprocal is

$$\begin{aligned} G_\mu^{'0} &= \cos\delta_{23}G_\mu^0 + \sin\delta_{23}B'_\mu \ , \\ W_\mu^{'0} &= -\sin\delta_{23}G_\mu^0 + \cos\delta_{23}B'_\mu \ . \end{aligned} \tag{4.10}$$

The remaining $U_1(1)$ group is a particular sub-group of $U_3(1) \times U_2(1)$ obtained through the mixing process. If we denote the elements of this $U_1(1)$ group by $s(x)$, the second scalar field, through which we reduce eventually the symmetry to that of hyper-charge, should transform as

$$\Phi_2(x) \to s(x)\Phi_2(x)v^{-1}(x) \tag{4.11}$$

(with no $\star$ product), and hence its covariant derivative, which only involves the θ-independent parts of the gauge fields, is given by

$$D_\mu = \partial_\mu + \frac{\mathrm{i}}{2}g_0B'_\mu - \frac{\mathrm{i}}{2}g_1B_\mu \ , \tag{4.12}$$

where $g_0 = 2g_2 3g_3/\sqrt{(2g_2)^2 + (3g_3)^2}$ is the coupling constant to the residual B'_μ field. Following exactly the same prescription as before for the Higgs mechanism (i.e. assuming the Lagrangian for the Φ_2 field to be similar to that of Φ_1, given by (4.6)), we shall end up with a new gauge boson, W_μ^0, whose mass term in the Lagrangian will read

$$\frac{1}{4}(g_0B'_\mu - g_1B_\mu)^2\phi_2^2 = N_2^2(W_\mu^0)^2\phi_2^2 \ , \tag{4.13}$$

where $N_2 = \frac{1}{2}\sqrt{g_0^2 + g_1^2}$ and ϕ_2 is the vacuum expectation value for Φ_2. The massive field, W_μ^0, is related to the fields B_μ, B'_μ through a rotation in the (B_μ, B'_μ) plane by the angle $\delta_{11'}$:

$$\begin{aligned} W_\mu^0 &= \cos\delta_{11'}B'_\mu - \sin\delta_{11'}B_\mu, \\ Y_\mu &= \sin\delta_{11'}B_\mu^{'0} + \cos\delta_{11'}B_\mu \ . \end{aligned} \tag{4.14}$$

The inverse of this transformation, which relates W^0 and Y (the hyper-photon field) to B' and B, is

$$\begin{aligned} B'_\mu &= \cos\delta_{11'}W_\mu^0 + \sin\delta_{11'}Y_\mu \ , \\ B_\mu &= -\sin\delta_{11'}W_\mu^0 + \cos\delta_{11'}Y_\mu \ . \end{aligned} \tag{4.15}$$

To summarize, we have reduced the three $U_n(1)$ factors to a single $U_1(1)$ through two proper Higgsac fields, Φ_1 and Φ_2 (in principle, in two different energy scales); in the end, instead of the corresponding three $U(1)$ fields, $G^{'0}$, $W^{'0}$ and B^0, we have introduced two massive gauge bosons, G^0 and W^0 and the (massless) hyper-photon Y. The initial and final $U(1)$ gauge fields are hence related by a 3×3 rotation matrix R:

$$\begin{pmatrix} G_\mu^{'0} \\ W_\mu^{'0} \\ B_\mu \end{pmatrix} = R_{3\times3} \begin{pmatrix} G_\mu^0 \\ W_\mu^0 \\ Y_\mu \end{pmatrix} \ , \tag{4.16}$$

where

$$R_{3\times3} = R_{23}\ R_{11'} \ ; \tag{4.17}$$

$$\begin{aligned} R_{23} &= \begin{pmatrix} \cos\delta_{23} & \sin\delta_{23} & 0 \\ -\sin\delta_{23} & \cos\delta_{23} & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \\ R_{11'} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_{11'} & \sin\delta_{11'} \\ 0 & -\sin\delta_{11'} & \cos\delta_{11'} \end{pmatrix} \ . \end{aligned} \tag{4.18}$$

It is clear from the form of (4.16) that it does not matter in which order we reduce the $U(1)$ symmetries.

The masses of the massive gauge bosons depend on the Φ_1 and Φ_2 vacuum expectation values:

$$\begin{aligned} m_{G^0} &= \sqrt{g_2^2 + \left(\frac{3}{2}g_3\right)^2}\ |\phi_1| \ , \\ m_{W^0} &= \sqrt{\left(\frac{1}{2}g_1\right)^2 + g_2^2 + \left(\frac{3}{2}g_3\right)^2}\ |\phi_2| \ . \end{aligned} \tag{4.19}$$

Then it is straightforward to rewrite the action (4.3) in terms of the physical gauge fields (G^a_μ, W^i_μ, Y_μ; G^0_μ, W^0_μ). We still need to define the Z and photon fields out of them, but we will not work this out here and we postpone it to the next section, where we discuss the electro-weak symmetry breaking. However, we would like to comment that, as we discussed in Sect. 3, the Lagrangian that one will find for the hyper-photon Y_μ (upon insertion of (4.16) into (4.3)) is *not* of the form specific for a pure $U_\star(1)$ theory.

After the two Higgsac reductions we end up with the NC($SU_c(3) \times SU_{\rm L}(2) \times U_Y(1)$), where similar to the NC(QED + QCD) case, this group is in fact a group which in the $\theta \to 0$ limit recovers the usual SM. However, for non-zero θ it receives some non-commutative corrections. In fact this algebra is the enveloping algebra of the usual $SU_c(3) \times SU_{\rm L}(2) \times U_Y(1)$ algebra defined by insertion of the star products.

4.2 The matter content

When coupling the matter fields to the $U_\star(3)\times U_\star(2)\times U_\star(1)$ theory, we have to keep in mind that, according to [1] (see (iii) in Sect. 2.1), since we have three simple factors in our group, we can have only $\frac{1}{2}3 \times (3+1) = 6$ types of charged particles, in the fundamental and/or anti-fundamental representation of the group factors. We note that the symmetry-reducing scalar particles (Higgsac fields) used in the previous subsection which have to be charged under $U(1)$ *factor-groups* of $U_\star(1)$, $U_\star(2)$ and $U_\star(3)$, are not included among these six types of particles.

Let us now give the gauge transformation properties of these matter fields, together with their corresponding covariant derivatives.

(1) Right-handed charged leptons (in anti-fundamental representation of $U_\star(1)$). In this group we consider the right-handed electron, which transforms as

$$e_{\rm R}(x) \ \to \ e_{\rm R}(x)\, v^{-1}(x) \ , \tag{4.20}$$

and hence the corresponding covariant derivative is

$$D^1_\mu e_{\rm R}(x) = \partial_\mu e_{\rm R}(x) - \frac{\rm i}{2} g_1 e_{\rm R}(x) B_\mu . \tag{4.21}$$

(2) Left-handed leptons (in the fundamental representation of $U_\star(2)$ and anti-fundamental representation of $U_\star(1)$). Here we shall include the left-handed electron and its neutrino, in a doublet:

$$\Psi^l_{\rm L}(x) = \begin{pmatrix} \nu(x) \\ e(x) \end{pmatrix}_{\rm L} . \tag{4.22}$$

Under the gauge transformations, the doublet transforms as

$$\Psi^l_{\rm L}(x) \ \to \ V(x)\, \Psi^l_{\rm L}(x)\, v^{-1}(x) \ , \tag{4.23}$$

and therefore the corresponding covariant derivative is

$$\begin{aligned} D^{1+2}_\mu \Psi^l_{\rm L}(x) &= \partial_\mu \Psi^l_{\rm L}(x) + \frac{\rm i}{2} g_2 W'^0_\mu \Psi^l_{\rm L}(x) \\ &\quad + \frac{\rm i}{2} g_2 W^i_\mu \sigma_i \Psi^l_{\rm L}(x) - \frac{\rm i}{2} g_1 \Psi^l_{\rm L}(x) B_\mu \ . \end{aligned} \tag{4.24}$$

(3,4) Right-handed quarks. Here, we choose the right-handed up quark in the fundamental representation of $U_\star(1)$ and anti-fundamental representation of $U_\star(3)$ and the right-handed down quark in the anti-fundamental representation of $U_\star(3)$:

$$\begin{aligned} u_{\rm R}(x) \ &\to \ v(x) u_{\rm R}(x)\, U^{-1}(x) \ , \\ d_{\rm R}(x) \ &\to \ d_{\rm R}(x)\, U^{-1}(x) \ , \end{aligned} \tag{4.25}$$

with the covariant derivatives

$$\begin{aligned} D^{1+3}_\mu u_{\rm R}(x) &= \partial_\mu u_{\rm R}(x) + \frac{\rm i}{2} g_1 B_\mu u_{\rm R}(x) - \frac{\rm i}{2} g_3 u_{\rm R}(x) G'^0_\mu \\ &\quad - \frac{\rm i}{2} g_3 u_{\rm R}(x) G^a_\mu T^a \ , \end{aligned} \tag{4.26}$$

$$D^3_\mu d_{\rm R}(x) = \partial_\mu d_{\rm R}(x) - \frac{\rm i}{2} g_3 d_{\rm R}(x) G'^0_\mu - \frac{\rm i}{2} g_3 d_{\rm R}(x) G^a_\mu T^a \ . \tag{4.27}$$

(5) Left-handed quarks. The doublet of left-handed up and down quarks,

$$\Psi^q_{\rm L}(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}_{\rm L} , \tag{4.28}$$

are in the fundamental representation of $U_\star(2)$ and anti-fundamental representation of $U_\star(3)$:

$$\Psi^q_{\rm L}(x) \ \to \ V(x)\, \Psi^q_{\rm L}(x)\, U^{-1}(x) \ , \tag{4.29}$$

with the covariant derivative

$$\begin{aligned} D^{2+3}_\mu \Psi^q_{\rm L}(x) &= \partial_\mu \Psi^q_{\rm L}(x) + \frac{\rm i}{2} g_2 W'^0_\mu \Psi^q_{\rm L}(x) + \frac{\rm i}{2} g_2 W^i_\mu \sigma_i \Psi^q_{\rm L}(x) \\ &\quad - \frac{\rm i}{2} g_3 \Psi^q_{\rm L}(x) G'^0_\mu - \frac{\rm i}{2} g_3 \Psi^q_{\rm L}(x) G^a_\mu T^a \ . \end{aligned} \tag{4.30}$$

(6) The Higgs doublet. We have

$$\Phi(x) = \begin{pmatrix} \Phi^+(x) \\ \Phi^0(x) \end{pmatrix} , \tag{4.31}$$

in the fundamental representation of $U_\star(2)$,

$$\Phi(x) \ \to \ V(x)\, \Phi(x) \ , \tag{4.32}$$

with the covariant derivative

$$D^2_\mu \Phi(x) = \partial_\mu \Phi(x) + \frac{\rm i}{2} g_2 W'^0_\mu \Phi(x) + \frac{\rm i}{2} g_2 W^i_\mu \sigma_i \Phi(x) \ . \tag{4.33}$$

We stress that the Higgs field interacts with other matter and gauge fields directly, i.e. in (4.32) and (4.33) we should use $\star$-products and the full gauge fields (not only their θ-independent parts, as is the case for Higgsac fields). We would also like to remark that the Higgs doublet fits perfectly in this picture and also exhausts the possible types of charged particles allowed by the no-go theorem [1].

Now, let us show how the $U_\star(1)$ symmetry reduction solves the hyper-charge quantization problem. This fact will be made obvious by showing that the coupling of all matter fields to the massless gauge field Y_μ of the residual $U_\star(1)$ is realized through the usual hyper-charges of the particles (see Table 1). To this end, we consider one by one the relevant terms of the Lagrangian (i.e. $\bar{\Psi}\gamma^\mu D_\mu\Psi$) for each type of matter field. In what follows, we denote the coupling constant to the hyper-charge $U_\star(1)$ by g'. The order in which we discuss the different types of fields is the most convenient one:

(i) Right-handed electron: the coupling to Y_μ can be read off by using (4.15):

$$\begin{aligned}\bar{e}_\mathrm{R}\gamma^\mu D^1_\mu e_\mathrm{R} &= \bar{e}_\mathrm{R}\gamma^\mu\partial_\mu e_\mathrm{R} - \frac{\mathrm{i}}{2}g_1\bar{e}_\mathrm{R}\gamma^\mu e_\mathrm{R}B_\mu \\ &= \bar{e}_\mathrm{R}\gamma^\mu\partial_\mu e_\mathrm{R} - \frac{\mathrm{i}}{2}g_1\cos\delta_{11'}\bar{e}_\mathrm{R}\gamma^\mu e_\mathrm{R}Y_\mu + \cdots ,\end{aligned} \tag{4.34}$$

where the dots contain the coupling to the massive gauge bosons, G^0_μ and W^0_μ. As the coupling term should be proportional to the hyper-charge of e_R, i.e. $-2g'$, we *define* g' by

$$\frac{1}{2}g_1\cos\delta_{11'} = g' . \tag{4.35}$$

Also, we note that using the definition of the $\delta_{11'}$ mixing angle (4.14) we have

$$\cot\delta_{11'} = \frac{3g_3\ 2g_2}{g_1\sqrt{(2g_2)^2+(3g_3)^2}} , \tag{4.36}$$

and hence

$$g' = \frac{1}{2}\frac{3g_3\ 2g_2}{\sqrt{(2g_2)^2+(3g_3)^2}}\ \sin\delta_{11'} . \tag{4.37}$$

(ii) Right-handed down quark: in the same way as above, using (4.27) and (4.16), we obtain

$$\begin{aligned}&\bar{d}_\mathrm{R}\gamma^\mu D^3_\mu d_\mathrm{R} \\ &= \bar{d}_\mathrm{R}\gamma^\mu\partial_\mu d_\mathrm{R} - \frac{\mathrm{i}}{2}g_3\bar{d}_\mathrm{R}\gamma^\mu d_\mathrm{R}G'^0_\mu - \frac{\mathrm{i}}{2}g_3\bar{d}_\mathrm{R}\gamma^\mu d_\mathrm{R}G^a_\mu T^a \\ &= \bar{d}_\mathrm{R}\gamma^\mu\partial_\mu d_\mathrm{R} - \frac{\mathrm{i}}{2}g_3\sin\delta_{23}\bar{d}_\mathrm{R}\gamma^\mu d_\mathrm{R}B'_\mu - \cdots \qquad (4.38) \\ &= \bar{d}_\mathrm{R}\gamma^\mu\partial_\mu d_\mathrm{R} - \frac{\mathrm{i}}{2}g_3\sin\delta_{23}\sin\delta_{11'}\bar{d}_\mathrm{R}\gamma^\mu d_\mathrm{R}Y_\mu - \cdots\end{aligned}$$

From this one can readily find the hyper-charge of the right-handed down quark, Y_{d_R}:

$$g_3\sin\delta_{23}\sin\delta_{11'} = -Y_{d_\mathrm{R}} . \tag{4.39}$$

However, using (4.36) we find that

$$Y_{d_\mathrm{R}} = -\frac{2}{3}g' , \tag{4.40}$$

which exactly yields the values given in Table 1. In the following, we show that all the other hyper-charges will also come out correctly, using (4.8), (4.35) and (4.36).

(iii) Right-handed up quark: similarly as before

$$\begin{aligned}&\bar{u}_\mathrm{R}\gamma^\mu D^{1+3}_\mu u_\mathrm{R} \\ &= \bar{u}_\mathrm{R}\gamma^\mu\partial_\mu u_\mathrm{R} + \frac{\mathrm{i}}{2}g_1\bar{u}_\mathrm{R}\gamma^\mu B_\mu u_\mathrm{R} - \frac{\mathrm{i}}{2}g_3\bar{u}_\mathrm{R}\gamma^\mu u_\mathrm{R}G'^0_\mu - \cdots \\ &= \bar{u}_\mathrm{R}\gamma^\mu\partial_\mu u_\mathrm{R} \\ &+ \frac{\mathrm{i}}{2}(g_1\cos\delta_{11'} - g_3\sin\delta_{23}\sin\delta_{11'})\bar{u}_\mathrm{R}\gamma^\mu Y_\mu u_\mathrm{R} \\ &- \frac{\mathrm{i}}{2}g_3\sin\delta_{23}\sin\delta_{11'}\bar{u}_\mathrm{R}\gamma^\mu[Y_\mu,u_\mathrm{R}]_\star - \cdots ,\end{aligned} \tag{4.41}$$

from which it emerges that

$$g_1\cos\delta_{11'} - g_3\sin\delta_{23}\sin\delta_{11'} = Y_{u_\mathrm{R}} , \tag{4.42}$$

where Y_{u_R} is the hyper-charge of the right-handed up quark. Using (4.35) and (4.36), we find

$$Y_{u_\mathrm{R}} = \frac{4}{3}g' . \tag{4.43}$$

(iv) Left-handed leptons: for the doublet of left-handed leptons, we find

$$\begin{aligned}&\bar{\Psi}^l_\mathrm{L}\gamma^\mu D^{1+2}_\mu\Psi^l_\mathrm{L} \\ &= \bar{\Psi}^l_\mathrm{L}\gamma^\mu\partial_\mu\Psi^l_\mathrm{L} + \frac{\mathrm{i}}{2}g_2\bar{\Psi}^l_\mathrm{L}\gamma^\mu W'^0_\mu\Psi^l_\mathrm{L} - \frac{\mathrm{i}}{2}g_1\bar{\Psi}^l_\mathrm{L}\gamma^\mu\Psi^l_\mathrm{L}B_\mu \\ &+ \cdots \\ &= \bar{\Psi}^l_\mathrm{L}\gamma^\mu\partial_\mu\Psi^l_\mathrm{L} \\ &- \frac{\mathrm{i}}{2}(g_1\cos\delta_{11'} - g_2\cos\delta_{23}\sin\delta_{11'})\bar{\Psi}^l_\mathrm{L}\gamma^\mu\Psi^l_\mathrm{L}Y_\mu \\ &- \frac{\mathrm{i}}{2}g_2\cos\delta_{23}\sin\delta_{11'}\bar{\Psi}^l_\mathrm{L}\gamma^\mu[\Psi^l_\mathrm{L},Y_\mu]_\star + \cdots ,\end{aligned} \tag{4.44}$$

from which we read off the condition

$$g_1\cos\delta_{11'} - g_2\cos\delta_{23}\sin\delta_{11'} = -Y_{\Psi^l_\mathrm{L}} . \tag{4.45}$$

Using (4.8), (4.35) and (4.39), from (4.45) we obtain

$$Y_{\Psi^l_\mathrm{L}} = -g' . \tag{4.46}$$

(v) Left-handed quarks: in this case, the relevant coupling term will read

$$\begin{aligned}&\bar{\Psi}^q_\mathrm{L}\gamma^\mu D^{2+3}_\mu\Psi^q_\mathrm{L} \\ &= \bar{\Psi}^q_\mathrm{L}\gamma^\mu\partial_\mu\Psi^q_\mathrm{L} + \frac{\mathrm{i}}{2}g_2\bar{\Psi}^q_\mathrm{L}\gamma^\mu W'^0_\mu\Psi^q_\mathrm{L} - \frac{\mathrm{i}}{2}g_3\bar{\Psi}^q_\mathrm{L}\gamma^\mu\Psi^q_\mathrm{L}G'^0_\mu \\ &+ \cdots \\ &= \bar{\Psi}^q_\mathrm{L}\gamma^\mu\partial_\mu\Psi^q_\mathrm{L} \\ &+ \frac{\mathrm{i}}{2}(g_2\cos\delta_{23} - g_3\sin\delta_{23})\sin\delta_{11'}\bar{\Psi}^q_\mathrm{L}\gamma^\mu Y_\mu\Psi^q_\mathrm{L} \\ &+ \frac{\mathrm{i}}{2}g_3\sin\delta_{23}\sin\delta_{11'}\bar{\Psi}^q_\mathrm{L}\gamma^\mu[Y_\mu,\Psi^q_\mathrm{L}]_\star + \cdots ,\end{aligned} \tag{4.47}$$

which implies that

$$(g_2\cos\delta_{23} - g_3\sin\delta_{23})\sin\delta_{11'} = Y_{\Psi^q_\mathrm{L}} , \tag{4.48}$$

and, recalling (4.8), (4.35) and (4.39), the hyper-charge of the left-handed quark doublet in units of g' is found to be

$$Y_{\Psi^q_\mathrm{L}} = \frac{1}{3}g' . \tag{4.49}$$

(vi) Higgs doublet: for the last one of the possible charged particles of our model,

$$\begin{aligned}&\bar{\Phi}\gamma^\mu D^2_\mu\Phi \\ &= \bar{\Phi}\gamma^\mu\partial_\mu\Phi + \frac{\mathrm{i}}{2}g_2\bar{\Phi}\gamma^\mu W'^0_\mu\Phi + \cdots \qquad (4.50) \\ &= \bar{\Phi}\gamma^\mu\partial_\mu\Phi + \frac{\mathrm{i}}{2}g_2\cos\delta_{23}\sin\delta_{11'}\bar{\Phi}\gamma^\mu Y_\mu\Phi + \cdots ,\end{aligned}$$

implying

$$g_2 \cos\delta_{23} \sin\delta_{11'} = Y_\Phi \ , \qquad (4.51)$$

and eventually, with the help of (4.8), (4.35) and (4.39),

$$Y_\Phi = g' \ . \qquad (4.52)$$

Before proceeding with the electro-weak symmetry breaking, let us recount the number of parameters that we have introduced. There are three different couplings, g_1, g_2 and g_3 which correspond to the $U_\star(1)$, $U_\star(2)$ and $U_\star(3)$ factors, respectively. In addition we have introduced two mixing angles, δ_{23} and $\delta_{11'}$. However, the physical couplings are g_2, the weak coupling, g_3, the strong coupling and $g' = \frac{1}{2} g_1 \cos\delta_{11'}$, the hyper-photon coupling. Also, there are two relations between the couplings and these mixing angles:

$$\tan\delta_{23} = \frac{2g_2}{3g_3} \ , \quad \sin\delta_{11'} = \frac{g'}{g_2}\sqrt{1+\left(\frac{2g_2}{3g_3}\right)^2} \ . \qquad (4.53)$$

Therefore, both of the mixing angles can be expressed in terms of the physical couplings g', g_2 and g_3.

Here we have chosen a specific order for the symmetry reductions and the Higgsac fields, namely, first we reduced the $U(1)$ of $U_\star(3) \times U_\star(2)$ and then the resulting $U(1)$ with the extra $U_\star(1)$ that we have in our gauge group. We would like to comment that the choice of any possible two Higgsac fields as well as the order of the symmetry reduction(s) do not change the charge assignments for the quarks and leptons. (Essentially these charges only depend on the representations of the particles and the fact that we start with the $U_\star(3) \times U_\star(2) \times U_\star(1)$ groups.)

5 The electroweak symmetry breaking

So far, starting from the $U_\star(3) \times U_\star(2) \times U_\star(1)$ gauge theory and reducing two $U(1)$ factors, we have arrived at a theory which can be called $\mathrm{NC}(SU_c(3) \times SU_\mathrm{L}(2) \times U(1))$. In order to complete the formulation of the NCSM, still we should proceed with the usual symmetry breaking through the Higgs doublet. In fact, by this symmetry breaking, fermions become massive through the Yukawa terms, which are also allowed in the non-commutative case. However, a more important role of this symmetry breaking is to give masses to the W^i_μ, $i = 1, 2, 3$ fields and also to define the massless photon and massive Z_μ through a combination of Y_μ and W^3_μ.

In this section, first we work out the details of this symmetry breaking in the gauge bosons sector and then in Sect. 5.2, we present the interaction terms of the fermions with the physical gauge bosons, as well as the corresponding Yukawa terms. We also compare these interaction terms with those of the usual standard model.

For performing the electro-weak symmetry breaking, we use a doublet of scalar fields of the type (4.31), charged under the $U_\star(2)$ symmetry group (before the reduction of the $U(1)$ factors of $U_\star(3)\times U_\star(2)\times U_\star(1)$). Practically, after the $U(1)$ symmetry reduction, this doublet would carry hyper-charge and weak charge. The new terms occurring in the full electro-weak Lagrangian before the symmetry breaking and due to the presence of the doublet of scalar fields are

$$\mathcal{L}_\mathrm{Higgs} = (D_\mu\Phi)^\dagger(D_\mu\Phi) + \mu^2\Phi^\dagger\Phi - \frac{f}{4!}(\Phi^\dagger\Phi)^2_\star + \mathcal{L}_\mathrm{Yukawa} \ , \qquad (5.1)$$

where

$$\begin{aligned}\mathcal{L}_\mathrm{Yukawa} &= h_e\bar{e}_\mathrm{R}\Phi^\dagger\Psi^l_\mathrm{L} + h_e^*\bar{\Psi}^l_\mathrm{L}\Phi e_\mathrm{R} \\ &+ h_d\bar{d}_\mathrm{R}\Phi^\dagger\Psi^q_\mathrm{L} + h_d^*\bar{\Psi}^q_\mathrm{L}\Phi d_\mathrm{R} \\ &+ h_u\bar{u}_\mathrm{R}(\Phi^\mathrm{c})^\dagger\Psi^q_\mathrm{L} + h_u^*\bar{\Psi}^q_\mathrm{L}\Phi^\mathrm{c} u_\mathrm{R} \ , \end{aligned} \qquad (5.2)$$

and h_e, h_d and h_u are the respective Yukawa couplings. In (5.2), Φ^c is the charge conjugated field of Φ, which transforms as

$$\Phi^\mathrm{c} \ \rightarrow \ V(x)\,\Phi^\mathrm{c}\,v^{-1} \ . \qquad (5.3)$$

Noting (4.32), the relation (5.3) may seem unusual. However, we recall that the non-commutative particles, besides the usual charge, also carry higher-pole charges and in particular the dipole charge. Therefore, the charge conjugate of any particle is a particle which is carrying the opposite of all these higher-pole charges, as well as the charge itself. In fact, one can check that the charge conjugate of Higgs, Φ^c, should transform as (5.3).

One should also note that it is not possible to construct the Yukawa terms in the Lagrangian corresponding to the $U(1)$ symmetry-reducing Higgsac fields Φ_1 and Φ_2, because no gauge-invariant combination of them with the fermionic fields could exist.

The potential of (5.1) has a minimum at $(\Phi^\dagger\Phi)_0 = \frac{12\mu^2}{f} \equiv \phi_0^2$ and we can choose the vacuum expectation value for the scalar field to be[7]

$$\Phi_0 = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} \ . \qquad (5.4)$$

5.1 Symmetry breaking in the gauge bosons sector

Now we discuss the details of the electro-weak symmetry breaking and its implications on the gauge bosons sector. To this end, we write the full covariant derivative of the Higgs field, which is the main ingredient of the mass-generating term. Having in view (4.10), (4.15) and (4.33), we obtain

$$\begin{aligned} D^2_\mu\Phi(x) &= \partial_\mu\Phi(x) \\ &+ \frac{\mathrm{i}}{2}\left[g'Y_\mu + \sin\delta_{23}\left(-g_2G^0_\mu + \frac{3}{2}g_3\cos\delta_{11'}W^0_\mu\right)\right]\Phi(x) \\ &+ \frac{\mathrm{i}}{2}g_2W^i_\mu\sigma_i\Phi(x) \ . \end{aligned} \qquad (5.5)$$

[7] We note that, since this minimum is x-independent, one can drop the $\star$-products and hence the minimum-energy solution is the same as in the commutative case

Hence, the mass term emerging is of the form of

$$L_M = \Big\{ \Big[(g' Y_\mu - g_2 W^3_\mu) + \sin\delta_{23} \left(-g_2 G^0_\mu + \frac{3}{2} g_3 \cos\delta_{11'} W^0_\mu \right) \Big]^2 + 2 g_2^2 W^+_\mu W^-_\mu \Big\} |\phi_0|^2 \,, \quad (5.6)$$

where $W^\pm = \frac{1}{\sqrt{2}}(W^1 \pm \mathrm{i} W^2)$.

Following the usual Higgs mechanism, let us identify the first term in the brackets in (5.6) as being proportional to Z^0_μ; more explicitly

$$Z^0_\mu = -\sin\theta^0_{\mathrm{W}} Y_\mu + \cos\theta^0_{\mathrm{W}} W^3_\mu \,, \quad (5.7)$$

$$A_\mu = \cos\theta^0_{\mathrm{W}} Y_\mu + \sin\theta^0_{\mathrm{W}} W^3_\mu \,, \quad (5.8)$$

where the weak-mixing angle θ^0_{W} is defined as in the usual standard model:

$$\tan\theta^0_{\mathrm{W}} = \frac{g'}{g_2} \,. \quad (5.9)$$

With these notations, we can rewrite (5.6) as

$$L_M = \Big\{ \Big[g Z^0_\mu + \sin\delta_{23} \left(-g_2 G^0_\mu + \frac{3}{2} g_3 \cos\delta_{11'} W^0_\mu \right) \Big]^2 + 2 g_2^2 W^+_\mu W^-_\mu \Big\} |\phi_0|^2 \,, \quad (5.10)$$

where

$$g = \sqrt{g'^2 + g_2^2} \,. \quad (5.11)$$

In (5.10), the mass term for the $W^\pm$ bosons is clearly singled out:

$$m_{W^\pm} = g \cos\theta^0_{\mathrm{W}} |\phi_0| \,. \quad (5.12)$$

As we see from (5.10), Z^0_μ is not the real physical Z-boson. The physical Z-particle, which diagonalizes the above mass Lagrangian, is mixed with the other two massive gauge bosons, W^0_μ and G^0_μ. As a result, we have a correction to the mass of the physical Z-particle, compared to the usual standard model. However, still the massless gauge boson, the photon, is given by (5.8).

In order to compute this correction, we have to take into account also the mass terms of W^0_μ and G^0_μ obtained during the $U(1)$ symmetry breaking and to diagonalize the obtained mass matrix. Recalling (4.19) and (5.11), we can write the mass term for Z^0_μ, W^0_μ and G^0_μ:

$$\begin{aligned} L_{ZWG} = \frac{g^2}{2} \Big\{ & \Big[Z^0_\mu + \cos\theta^0_{\mathrm{W}} \Big(\sin\delta_{23} G^0_\mu - \cos\delta_{11'} \cos\delta_{23} W^0_\mu \Big) \Big]^2 |\phi_0|^2 + \left(\frac{\cos^2\theta^0_{\mathrm{W}}}{\sin^2\delta_{23}} \phi_1^2 \right) G^{02}_\mu \\ & + \left(\frac{\cos^2\theta^0_{\mathrm{W}} \cos^2\delta_{23}}{\cos^2\delta_{11'}} \phi_2^2 \right) W^{02}_\mu \Big\} \\ \equiv {} & \frac{g^2}{2} \phi_0^2 \, \mathbf{X}^{\mathbf{t}} \mathbf{M} \mathbf{X} \,, \end{aligned} \quad (5.13)$$

where

$$\mathbf{X} = \begin{pmatrix} Z^0_\mu \\ G^0_\mu \\ W^0_\mu \end{pmatrix} \,, \qquad \mathbf{M} = \begin{pmatrix} 1 & a & -b \\ a & a^2 + d^2 & -ab \\ -b & -ab & f^2 + b^2 \end{pmatrix} \,, \quad (5.14)$$

with

$$a = \cos\theta^0_{\mathrm{W}} \sin\delta_{23} \,, \quad b = \cos\theta^0_{\mathrm{W}} \cos\delta_{11'} \cos\delta_{23} \,,$$
$$d = \frac{\cos\theta^0_{\mathrm{W}}}{\sin\delta_{23}} \frac{\phi_1}{|\phi_0|} \,, \quad f = \frac{\cos\theta^0_{\mathrm{W}} \cos\delta_{23}}{\cos\delta_{11'}} \frac{\phi_2}{|\phi_0|} \,. \quad (5.15)$$

Since the physical Z-field, Z_μ, and Z^0_μ should almost be equivalent, we expect the a and b factors of (5.14) to be small (compared to d and f). Physically, this is equivalent to assuming that

$$\frac{m_Z}{m_{G^0}}, \ \frac{m_Z}{m_{W^0}} \ll 1 \,.$$

Then, diagonalizing (5.14), the mass for the physical Z-particle, up to the second order in $\frac{m_Z}{m_{G^0}}$, $\frac{m_Z}{m_{W^0}}$ is found to be

$$m_Z^2 = g^2 |\phi_0|^2 \left[1 - \sin^4\delta_{23} \left(\frac{\phi_0}{\phi_1} \right)^2 - \cos^4\delta_{11'} \left(\frac{\phi_0}{\phi_2} \right)^2 \right] \,, \quad (5.16)$$

and therefore,

$$\frac{m_W^2}{m_Z^2} = \cos^2\theta^0_{\mathrm{W}} \left\{ 1 + \cos^2\theta^0_{\mathrm{W}} \left[\left(\frac{m_Z}{m_{G^0}} \right)^2 \sin^2\delta_{23} + \left(\frac{m_Z}{m_{W^0}} \right)^2 \cos^2\delta_{23} \cos^2\delta_{11'} \right] \right\} \,. \quad (5.17)$$

We also note that, using (4.53) and (5.9), we have $\cos^2\delta_{23} \cos^2\delta_{11'} = \cos^2\delta_{23} - \tan^2\theta^0_{\mathrm{W}}$.

Having identified the physical gauge fields: the massless gluons, G^a_μ, $a = 1, 2, \cdots, 8$; the photon, A_μ; the massive gauge bosons $W^\pm_\mu, Z_\mu, W^0_\mu$ and G^0_μ, one can rewrite the action (4.3) in terms of these fields and the corresponding couplings. Although we do not write the latter down here explicitly, we would like to comment that in the non-commutative case we have three- and four-photon interaction vertices, which are *not* the vertices arising from a pure $U_\star(1)$ theory. Besides the differences in the photon–photon vertices, there exist also photon–Z interaction terms which have no counter-part in the standard model.

5.2 Symmetry breaking in the fermionic sector

In order to pick up the fermionic interaction terms after the electro-weak symmetry breaking, we shall explicitly write down the relevant interaction terms of the $U_\star(3) \times U_\star(2) \times U_\star(1)$ Lagrangian separately for the leptonic and quark sectors.

For the leptonic sector, using (4.21) and (4.22), we have

$$\begin{aligned}\mathcal{L}_{\text{leptons}} = &-\frac{\mathrm{i}}{2} g_1 \bar{e}_{\mathrm{R}} \gamma^\mu e_{\mathrm{R}} B_\mu + \frac{\mathrm{i}}{2} g_2 \bar{\Psi}^l_{\mathrm{L}} \gamma^\mu W^i_\mu \sigma_i \Psi^l_{\mathrm{L}} \\ &+ \frac{\mathrm{i}}{2} \bar{\Psi}^l_{\mathrm{L}} \gamma^\mu (g_2 W^{'0}_\mu - g_1 B_\mu) \Psi^l_{\mathrm{L}} \\ &- \frac{\mathrm{i}}{2} g_1 \bar{\Psi}^l_{\mathrm{L}} \gamma^\mu [\Psi^l_{\mathrm{L}}, B_\mu]_\star \ , \end{aligned} \tag{5.18}$$

and for the quark sector, recalling (4.26), (4.27) and (4.30),

$$\begin{aligned}\mathcal{L}_{\text{quarks}} = &\frac{\mathrm{i}}{2} g_1 \bar{u}_{\mathrm{R}} \gamma^\mu B_\mu u_{\mathrm{R}} - \frac{\mathrm{i}}{2} g_3 \bar{u}_{\mathrm{R}} \gamma^\mu u_{\mathrm{R}} G^{'0}_\mu \\ &- \frac{\mathrm{i}}{2} g_3 \bar{u}_{\mathrm{R}} \gamma^\mu u_{\mathrm{R}} G^a_\mu T^a \\ &- \frac{\mathrm{i}}{2} g_3 \bar{d}_{\mathrm{R}} \gamma^\mu d_{\mathrm{R}} G^{'0}_\mu - \frac{\mathrm{i}}{2} g_3 \bar{d}_{\mathrm{R}} \gamma^\mu d_{\mathrm{R}} G^a_\mu T^a \\ &+ \frac{\mathrm{i}}{2} g_2 \bar{\Psi}^q_{\mathrm{L}} \gamma^\mu W^{'0}_\mu \Psi^q_{\mathrm{L}} + \frac{\mathrm{i}}{2} g_2 \bar{\Psi}^q_{\mathrm{L}} \gamma^\mu W^i_\mu \sigma_i \Psi^q_{\mathrm{L}} \\ &- \frac{\mathrm{i}}{2} g_3 \bar{\Psi}^q_{\mathrm{L}} \gamma^\mu \Psi^q_{\mathrm{L}} G^{'0}_\mu - \frac{\mathrm{i}}{2} g_3 \bar{\Psi}^q_{\mathrm{L}} \gamma^\mu \Psi^q_{\mathrm{L}} G^a_\mu T^a \ . \end{aligned} \tag{5.19}$$

After the reduction of the $U(1)$ factors and the electroweak symmetry breaking, from (5.18) and (5.19) we obtain the following interaction terms.

Leptonic sector. The electron–photon interaction vertex comes in a form analogous to that of the usual standard model:

$$\mathcal{L}_{\Psi_e-\gamma} = -\mathrm{i}e \bar{\Psi}_e \gamma^\mu \Psi_e A_\mu \ , \tag{5.20}$$

with e being the coupling. Following the symmetry-breaking procedure, we obtain

$$e = g_2 \sin\theta^0_{\mathrm{W}} = \frac{1}{2} g \ \sin 2\theta^0_{\mathrm{W}} \ , \tag{5.21}$$

using (5.9) and (5.11). From this form of the interaction term, it is clear that the electron is in the anti-fundamental representation of the residual $U_\star(1)$ group, described by the massless gauge field A_μ and corresponding to NCQED. For the electron–Z_μ vertex we have

$$\begin{aligned}\mathcal{L}_{\Psi_e-Z^0_\mu} = &\mathrm{i}g \left[\left(-\frac{1}{2} + \sin^2\theta^0_{\mathrm{W}} \right) \bar{\Psi}_{e_{\mathrm{L}}} \gamma^\mu Z^0_\mu \Psi_{e_{\mathrm{L}}} \right] \\ &+ \mathrm{i}g \sin^2\theta^0_{\mathrm{W}} \bar{\Psi}_{e_{\mathrm{R}}} \gamma^\mu Z^0_\mu \Psi_{e_{\mathrm{R}}} \\ &+ \mathrm{i}g \sin^2\theta^0_{\mathrm{W}} \bar{\Psi}_e \gamma^\mu [\Psi_e, Z^0_\mu]_\star \ , \end{aligned} \tag{5.22}$$

where the first two terms are of the same form as in the usual standard model. However, still one should keep the following in mind.

(1) Actually what appears in the interaction terms (5.22) is Z^0_μ and not the physical Z-particle. Hence, these terms also generate extra interaction terms between electron and G^0 and W^0 massive gauge bosons.

(2) Still one should be careful with the order of the fields, due to the $\star$-product.

In particular, we note the Moyal bracket term; indicating that the non-commutative electron besides the usual Z charge also couples to the derivatives of $Z-\mu$. In the first order in $\theta_{\mu\nu}$, basically this is a weak-dipole–Z interaction. For the electron–neutrino–$W^\pm_\mu$ interaction term we have

$$\mathcal{L}_{\Psi_e-\nu-W^\pm_\mu} = \mathrm{i}\sqrt{2} g_2 (\bar{\nu} \gamma^\mu W^+_\mu \Psi_{e_{\mathrm{L}}} + \bar{\Psi}_{e_{\mathrm{L}}} \gamma^\mu W^-_\mu \nu) \ , \tag{5.23}$$

which, apart from the $\star$-products between the fields, is the same as that of the usual neutrino–photon interaction.

For this we have

$$\mathcal{L}_{\nu-\gamma} = -\mathrm{i}e \bar{\nu} \gamma^\mu [\nu, A_\mu]_\star \ , \tag{5.24}$$

which is a completely new interaction, realized through the neutrino dipole moment. We will elaborate more on this interaction term and its physical consequences in the next section.

For the neutrino–Z_μ interaction we have

$$\mathcal{L}_{\nu-Z^0_\mu} = \frac{\mathrm{i}}{2} g \bar{\nu} \gamma^\mu Z^0_\mu \nu + \mathrm{i}g \sin^2\theta^0_{\mathrm{W}} \bar{\nu} \gamma^\mu [\nu, Z^0_\mu]_\star \ , \tag{5.25}$$

where the first term is of the same form as in the standard model. However, the second term is a result of the fact that the non-commutative neutrino also carries Z-dipole moment.

Quark sector. For the up quark–photon interaction we have

$$\mathcal{L}_{u-\gamma} = \frac{2\mathrm{i}}{3} e \bar{u} \gamma^\mu A_\mu u - \frac{\mathrm{i}}{3} e \bar{u} \gamma^\mu [u, A_\mu]_\star \ . \tag{5.26}$$

As we see, the up quark besides a simple insertion of the $\star$-product also involves another Moyal bracket term. This extra term which is basically coming from the fact that the up quark is non-singlet under the two group factors (4.25) and (4.29), has an interesting consequence: the electric dipole moment of the up quark is twice more than what is expected from naive NCQED. To see this let us expand (5.26) in powers of $\theta_{\mu\nu}$. Up to the first order we have

$$\mathcal{L}_{u-\gamma} = \frac{2\mathrm{i}}{3} e \bar{u} \gamma^\mu A_\mu u - \frac{2}{3} e \bar{u} \gamma^\mu \left(\theta_{\alpha\beta} \partial_\alpha A_\mu \partial_\beta u \right) + \mathcal{O}(\theta^2) \ . \tag{5.27}$$

Recalling the arguments of [21], one expects to find $\frac{1}{3}$ for the coefficient of the second term, while what we obtain is $\frac{2}{3}$.

For the down quark–photon interaction we have

$$\mathcal{L}_{d-\gamma} = -\frac{\mathrm{i}}{3} e \bar{d} \gamma^\mu \mathrm{d} A_\mu \ . \tag{5.28}$$

This is exactly what one expects from a naive extension of QED to NCQED, by insertion of $\star$-products.

For the up quark–Z_μ interaction we have

$$\begin{aligned}\mathcal{L}_{u-Z^0_\mu} = &\mathrm{i}g \left(\frac{1}{2} - \frac{2}{3} \sin^2\theta^0_{\mathrm{W}} \right) \bar{u}_{\mathrm{L}} \gamma^\mu Z^0_\mu u_{\mathrm{L}} \\ &- \mathrm{i}g \frac{2}{3} \sin^2\theta^0_{\mathrm{W}} \bar{u}_{\mathrm{R}} \gamma^\mu Z^0_\mu u_{\mathrm{R}} \\ &+ \mathrm{i}g \frac{1}{3} \sin^2\theta^0_{\mathrm{W}} \bar{u} \gamma^\mu [u, Z^0_\mu]_\star \ . \end{aligned} \tag{5.29}$$

Up to the difference in the $\star$-products (not written explicitly according to our convention), the first two terms are the same as in the usual standard model, while the third term is again showing the weak higher-pole moments of the non-commutative up quark.

For the down quark–Z_μ vertex we have

$$\begin{aligned}\mathcal{L}_{d-Z^0_\mu} &= \mathrm{i}g\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta^0_{\mathrm{W}}\right)\bar{d}_{\mathrm{L}}\gamma^\mu Z^0_\mu d_{\mathrm{L}}\\ &+\mathrm{i}g\frac{1}{3}\sin^2\theta^0_{\mathrm{W}}\bar{d}_{\mathrm{R}}\gamma^\mu Z^0_\mu d_{\mathrm{R}}\\ &+\mathrm{i}g\frac{1}{3}\sin^2\theta^0_{\mathrm{W}}\bar{d}\gamma^\mu[d,Z^0_\mu]_\star\ . \end{aligned} \tag{5.30}$$

For the up quark–down quark–$W^\pm_\mu$ interaction we have

$$\mathcal{L}_{u-d-W^\pm_\mu} = \frac{\mathrm{i}}{\sqrt{2}}g_2(\bar{u}_{\mathrm{L}}\gamma^\mu W^+_\mu d_{\mathrm{L}}+\bar{d}_{\mathrm{L}}\gamma^\mu W^-_\mu u_{\mathrm{L}})\ . \tag{5.31}$$

6 Some specific features of NCSM

In the previous section, we worked out in detail the fermions–gauge bosons interaction terms in the NCSM. In general, one can classify the new ingredients of the NCSM in two sets. First there are those coming from the group theoretical structure of the model and which do not depend on the non-commutativity parameter explicitly. This set is mainly a consequence of having two extra massive gauge bosons, G^0_μ and W^0_μ. Although we did not present it, almost all the fermions interact with the new massive gauge bosons, G^0_μ and W^0_μ. Such interaction terms effectively will give rise to Fermi's four-fermion interaction, where its coupling (up to some numeric factors) is $G_{\mathrm{F}}\left(\frac{m_Z}{m_{W^0}}\right)^2$. Another important effect of these new massive gauge bosons is the correction to the physical Z-particle, and in particular to its mass. We will discuss this in detail in Sect. 6.2 and in this way we impose some lower bounds on the masses of these new massive gauge bosons.

The second class of new features in the NCSM are the interaction terms coming from the $\star$-product and (at least at the classical level) in the commutative limit, i.e. $\theta\to 0$, they vanish explicitly. In other words, all the particles, besides the usual charge, up to the first order in $\theta_{\mu\nu}$, also carry a dipole charge which is proportional to the non-commutativity parameter [21,31]. From these new interaction terms here we discuss that of the neutrino–photon coupling and from this we obtain a lower bound on the non-commutativity scale.

6.1 Neutrino dipole moment

As we have explicitly shown in the previous section, and in particular in (5.24), the neutrino in the NCSM undergoes a new type of interaction: the neutrino–photon vertex. Unlike all the other photon–fermion interactions in the NCSM, this vertex is a *chiral* one, i.e. the only existing neutrino, the left-handed ν, appears in this interaction term. More precisely, in the non-commutative case, we do not necessarily need a right-handed neutrino to have a coupling to the electro-magnetic field, and therefore the neutrino, without being massive, can carry dipole charges.

On the other hand, there are very strong (astrophysical) bounds on the neutrino–photon interactions and especially the neutrino dipole moment [35]. In fact, these bounds can be translated to a lower bound on the non-commutativity scale, Λ_{NC}, defined by

$$\theta_{\mu\nu} = \frac{1}{\Lambda^2_{\mathrm{NC}}}\epsilon_{\mu\nu}\ , \tag{6.1}$$

where $\epsilon_{\mu\nu}$ is a dimensionless anti-symmetric parameter, whose elements are of the order of one.

It is well known that the neutrino has a considerable effect in the stellar cooling process. However, according to the standard model, they only participate in the weak interactions through the massive $W^\pm$ and Z. In this way any direct photon–neutrino interaction such as the one we have here can speed up the cooling process, which in turn will change the whole stellar evolution. To avoid drastic changes in this respect (which have not been observed) the strength of the neutrino–photon interaction should be smaller compared to that of Z. To materialize the above argument, let us expand (5.24) up to the first order in $\theta_{\mu\nu}$:

$$\mathcal{L}_{\nu-\gamma} = -\mathrm{i}e\bar{\nu}\gamma^\mu[\nu,A_\mu]_\star = -e\,\bar{\nu}\gamma^\mu\left(\theta_{\alpha\beta}\partial_\alpha A_\mu\partial_\beta\nu\right)+\mathcal{O}(\theta^2)\ . \tag{6.2}$$

As we see, in the above interaction the derivative of the neutrino appears (as well as that of the photon field A_μ). Then, one can read off the effective neutrino non-commutative dipole moment:

$$d_\nu = e\frac{1}{\Lambda^2_{\mathrm{NC}}}\ E_\nu\ , \tag{6.3}$$

where E_ν is the energy of the neutrino. For the case at hand, the solar neutrino problem, $E_\nu\simeq 10\,\mathrm{MeV}$ and the corresponding bound on the magnitude of the dipole moment is [32]

$$d_\nu \lesssim 0.1\times 10^{-10}\ \mu_{\mathrm{B}}\ , \tag{6.4}$$

where $\mu_{\mathrm{B}}=\frac{e\hbar}{2m_e c}$ is the Bohr magneton[8]. Therefore, one can readily obtain the lower bound on the non-commutativity scale

$$\Lambda_{\mathrm{NC}} \gtrsim 10^3\,\mathrm{GeV}\ . \tag{6.5}$$

Of course, this bound is based on a rough estimate and a more detailed calculation and a survey can improve this bound. Also we note that this bound is of the same order as the previous bounds coming from the Lamb shift [21] and the Lorentz-violation considerations [25].

[8] In fact this bound is coming from consideration of the red giant star's cooling process. There are some weaker and also stronger bounds on the neutrino dipole moment coming from some other sources. Since in our model we do not have right-handed neutrinos we cannot use the stronger bound of $10^{-13}\mu_{\mathrm{B}}$

Table 2. Standard model predictions for ϵ variables, at $m_H = 113\,\mathrm{GeV}$

$\epsilon_i \times 10^{+3}$	$m_t = 174.3 - 5.1$	$m_t = 174.3$	$m_t = 174.3 + 5.1$
ϵ_1	5.1	5.6	6.0
ϵ_2	−7.2	−7.4	−7.6
ϵ_3	5.4	5.4	5.3

6.2 Corrections to the weak-mixing angle

As we have discussed previously in Sect. 5, the physical Z-particle, which is an eigen-state of the mass matrix after the electro-weak symmetry breaking, besides the W^3_μ and the hyper-photon Y_μ, now receives a contribution from the other two new massive gauge bosons, G^0_μ and W^0_μ, while the photon field is only made out of W^3_μ and Y_μ, in such a way that at the end Z_μ and the photon field A_μ are orthonormal states. However, as we have explicitly shown, these contributions are suppressed by the $\left(\frac{m_Z}{m_{W^0}}\right)^2$ ratio, (5.16). On the other hand, the $W^\pm$ gauge bosons remain the same as in the usual standard model, $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \pm \mathrm{i}W^2_\mu)$. Therefore, the $\frac{m_Z}{m_W}$ ratio now receives a correction, as indicated in (5.17). We remind the reader that the weak-mixing angle θ^0_W is still defined through the ratio of hyper-photon coupling and weak coupling: $\frac{g'}{g_2} = \tan\theta^0_\mathrm{W}$.

In the usual standard model, although the parameter

$$\rho = \left(\frac{m_Z}{m_W}\right)^2 \cos^2\theta^0_\mathrm{W}$$

at the classical (tree) level is equal to one, it receives quantum (loop) corrections; see e.g. [33]. In fact, one of the precision tests of the standard model is to evaluate these corrections to ρ and compare them to the corresponding experimental data [33,34]. Here we use the conventions and notations of [33] to parameterize these corrections:

$$\left(\frac{m_Z}{m_W}\right)^2 = \left(\frac{m_Z}{m_W}\right)^2\Bigg|_B (1 + 1.43\epsilon_1 - 1.00\epsilon_2 - 0.86\epsilon_3)\ , \tag{6.6}$$

where the ϵ_i show the "large" asymptotic contributions, up to the leading linearized approximation and

$$\left(\frac{m_Z}{m_W}\right)^2\Bigg|_\mathrm{B} = 0.768905$$

is the Z and W mass ratio in the Born approximation. With the latest data used in [34], the *predicted* values of the ϵ variables in the usual standard model, which do depend on Higgs and top quark masses, are given in Table 2. However, the observed values of the ϵ_i obtained from all combined hadronic, leptonic and Higgs measurements are

$$\begin{aligned}\epsilon_1 &= (5.4 \pm 1.0) \times 10^{-3}\ ,\\ \epsilon_2 &= (-9.7 \pm 1.2) \times 10^{-3}\ ,\\ \epsilon_3 &= (5.4 \pm 0.9) \times 10^{-3}\ .\end{aligned} \tag{6.7}$$

Comparing the standard model results and the observed values (6.7), the non-commutative corrections should be smaller than the difference between these two values. More explicitly,

$$\cos^2\theta^0_\mathrm{W}\left[\left(\frac{m_Z}{m_{G^0}}\right)^2 \sin^2\delta_{23} + \left(\frac{m_Z}{m_{W^0}}\right)^2 \cos^2\delta_{23}\cos^2\delta_{11'}\right] \lesssim (2.014 \pm 3.404) \times 10^{-3}\ . \tag{6.8}$$

On the other hand,

$$\tan\delta_{23} = \frac{2}{3}\sqrt{\frac{\alpha_\mathrm{QED}}{\alpha_\mathrm{s}}\frac{1}{\sin^2\theta^0_\mathrm{W}}}\ \Bigg|_{m_Z} = 0.354\ , \tag{6.9}$$

where in the above we have used the data given in [35][9]. Now, if we assume that $m_{G^0} \simeq m_{W^0}$, we can find a lower bound on m_{G^0}:

$$m_{G^0} \gtrsim 2.5 \times 10\ m_Z\ . \tag{6.10}$$

7 Outlook

In this work we have constructed the non-commutative version of the standard model (NCSM). Mainly, the present article is devoted to presenting the formulation in which the obstacle against such a non-commutative version of the standard model has been overcome. We have classified these problems and obstacles in three categories; however, the most important one was the charge quantization problem. We have discussed how this problem can be resolved, while respecting the no-go theorem stating that matter fields cannot carry more than two kinds of charges [1]. In fact, as we have shown, *only* the matter content as in the usual standard model (including Higgs) is allowed in a non-commutative extension. Our recipe to remove these problems is based on the reduction of the extra $U(1)$ symmetries through the Higgs mechanism (and Higgsac fields), in which the residual massless $U(1)$ field becomes a linear combination of the original $U(1)$ fields, (4.16). The detailed discussion of the results in the 0-order in θ, compatible with the observed values for (hyper-) charges, is given in the appendices. We postpone the complete solution (to all orders in θ) of the $U(1)$ sub-groups reduction to a future work.

Actually, here we have just introduced the NCSM at the *classical level* and mainly in the *leading order in* θ and we have not explored all the possible new features of the NCSM. These are open questions to be studied in future works. However, among the new features we have briefly discussed, the neutrino dipole moment is a natural out-come of our model. This dipole moment interaction imposes a lower bound on the non-commutativity scale:

$$\Lambda_\mathrm{NC} \gtrsim 10^3\,\mathrm{GeV}\ . \tag{7.1}$$

[9] Using the relations defining $\delta_{11'}$ we find that $\sin^2\delta_{11'} = \frac{\tan^2\theta^0_\mathrm{W}}{\cos^2\delta_{23}} = 0.3383$

We have discussed that there are corrections to the $\frac{m_W^2}{m_Z^2}$ ratio, which do depend on the masses of the extra gauge bosons. Using the experimental bounds [33,34], we have found the lower bound on the masses of these gauge bosons:

$$m_{W^0},\ m_{G^0} \gtrsim 25\ m_Z\ . \qquad (7.2)$$

As we see, the bounds on all the three dimensionful parameters of our theory, $\Lambda_{\rm NC}, m_{W^0}$ and m_{G^0} are of the order of 1–10 TeV.

The other direct consequence of our model is the inherent CP violation, which is in both the leptonic (including neutrinos) and quark sectors and is controlled by the non-commutativity parameter, $\theta_{\mu\nu}$.

On the NCSM, there are a few other remarks in order.

(1) Most of our arguments for constructing the model in Sects. 3, 4 and 5 do not depend on the details of the $\star$-product we have used and only having a non-commutative (but associative) product would lead to the same conclusion.

(2) On anomaly cancellation: it is well known that an important theoretical consistency check for the usual standard model (as a chiral gauge theory) is the cancellation of the triangle anomaly. In fact, this anomaly cancellation is a consequence of the details of matter content and corresponding charges. In the non-commutative case, the anomaly calculations have been already done in [36]. According to these works a non-commutative gauge theory, in order to be anomaly free, should be vector-like. Hence, a non-commutative version of the standard model is incurably sick. However, along with the arguments of [37], the mixed anomalies (those which are of the form of $U_\star(n) - U_\star(n) - U_\star(m)$, $m \neq n$ and also $U_\star(1) - U_\star(2) - U_\star(3)$) are not present. Furthermore, our theory in the $U_\star(3)$ sector is vector-like. Although it is not clear how, we believe that the other two anomalous diagrams ($(U_\star(1))^3$ and $(U_\star(2))^3$) can be removed. One possible way, among others, as discussed in [37] can be making the supersymmetric version of NCSM. We hope that using the effective NC$SU(n)$ groups defined here we can solve the anomaly problem. We postpone a full analysis of the anomaly problem to future works.

(3) On quarks mixings: although we have not considered them here, the usual quarks mixings are also possible in the NCSM. If we only consider the usual unitary CKM mixing matrix (whose entries are constant and not space-time functions), the non-commutative effects will appear only at the loop level. (The non-commutativity appears as some overall phases in the amplitudes and hence in the probability and cross sections it will disappear.)

(4) On neutrino mass and mixing: in our model, neutrinos are massless, however, we can add masses and mixings. According to the no-go theorem, since we have exhausted all six possibilities for particles carrying any kind of charge, we cannot have a right-handed neutrino which carries a charge. Hence, the right-handed neutrino could only be a sterile neutrino, i.e. a singlet under all the $U_\star(1)$, $U_\star(2)$ and $U_\star(3)$ factors and could appear only through the mixing with active neutrinos, or it could be a dipole of one of the group factors, among which the most plausible is the $U_\star(1)$ factor, i.e. $\nu_{\rm R} \to v \nu_{\rm R} v^{-1}$.

Finally, as an immediate check for our model, one should examine the running of the non-commutative photon coupling, and as we have discussed, there is a reasonable hope to resolve the negative β-function problem of NCQED mentioned in [2].

Note added: After this paper was submitted to the hep-archive (hep-th/0107055), another very interesting work with the same main subject, by X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgennant has appeared [38]. In this work and also its follow-ups [39], the construction of the NCSM is based on the Seiberg–Witten map and it essentially differs from our approach in the fact that the internal symmetries are considered at the level of the algebra, while in our case they are considered at the gauge-group level. It is indeed very interesting to find and account for the different effects emerging from these two different approaches.

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Appendix

A Normal sub-groups of $U_\star(n)$

Let $\mathcal{A}_\star$ be the Lie algebra of functions on the Moyal plane generated by the commutators. For any $k = 0, 1, \ldots$, we define recursively

$$\mathcal{A}^\star_{k+1} \ = \ \{\mathcal{A}^\star_k, \mathcal{A}_\star\}_{MB}\ ,\ \mathcal{A}^\star_0 \ \equiv \ \mathcal{A}_\star\ . \qquad (A.1)$$

Any $f(x) \in \mathcal{A}_\star$ is a power series in $\theta_{\mu\nu}$; the set $\mathcal{A}^\star_k$ is formed by power series *starting* with the kth power. The set of sub-algebras $\mathcal{A}^\star_k$, $k = 0, 1, \ldots$, form a *filtration* of the Moyal plane: $\mathcal{A}^k_\star \mathcal{A}^m_\star \subset \mathcal{A}^{k+m}_\star$.

The gauge algebra $\mathcal{B} = u_\star(n)$ is defined as the following set of matrix functions on $\mathcal{A}_\star$:

$$\epsilon(x) \ = \ \epsilon_0(x)\mathbf{1}_n + \epsilon_a(x)T^a \ \equiv \ \epsilon_A(x)T^A\ , \qquad (A.2)$$

where $T^0 = \mathbf{1}_n$ is the $n \times n$ unit matrix and T^a, $a = 1, \ldots, n^2 - 1$, are $n \times n$ Gell-Mann matrices satisfying the relations $T^A T^B = \delta^{AB}\mathbf{1}_n + (d^{AB}_C + {\rm i} f^{AB}_C)T^C$; the $\epsilon_a(x) = \epsilon^\dagger_a(x)$ are hermitian functions belonging to the algebra $\mathcal{A}_\star = \mathcal{A}^\star_0$.

Let us consider now the commutator algebra $\mathcal{B}' = [\mathcal{B}, \mathcal{B}]$. This is an ideal formed by elements of $u_\star(n)$ with $\epsilon_0(x) \in \mathcal{A}^\star_1$ and $\epsilon_A(x) \in \mathcal{A}_\star$. Further, $\mathcal{B}'' = [\mathcal{B}', \mathcal{B}'] = \mathcal{B}'$. These properties of $\mathcal{B} = u_\star(n)$ and $\mathcal{B}' = {\rm NC}su(n)$ are analogous to those valid for the commutative gauge algebras $u(n)$ and $su(n)$.

Let us note that the sub-algebra $u_\star(1)$ of elements (A.2) with $\epsilon(x)=\epsilon_0(x)\mathbf{1}_n$ extends to another ideal $u^n_\star(1)$ by adding the $[u_\star(1),u_\star(n)]$ functions, i.e. $u^n_\star(1)$ is formed by the elements (A.2) with $\epsilon_0(x)\in\mathcal{A}_\star$ and $\epsilon_A(x)\in\mathcal{A}^\star_1$. Similarly, $\mathbf{1}^n_\star\equiv \mathrm{NC}su_n(1)=[u^n_\star(1),u^n_\star(1)]$ is an ideal in $u_\star(n)$.

With any ideal in $u_\star(n)$, we can associate the corresponding *factor-algebra*:

$$u_n(1)\ :=u_\star(n)/\mathrm{NC}su(n)\ , su(n)\ :=\ u_\star(n)/u^n_\star(1)\ . \tag{A.3}$$

The ideal $u_n(1)$ is formed by equivalency classes: $\epsilon(x)\sim\epsilon'(x)$ if $\epsilon(x)-\epsilon'(x)\in\mathrm{NC}su(n)$. However, any element of $u_\star(n)$ can be uniquely written in the form $\epsilon(x)=\epsilon^0(x)\mathbf{1}_n+\delta^1(x)$ with θ-independent $\epsilon^0(x)\in\mathcal{A}$ (here $\mathcal{A}\equiv\mathcal{A}_\star\setminus\mathcal{A}^\star_1$ is the factor-algebra isomorphic to the commutative algebra of functions) and $\delta^1(x)\in\mathrm{NC}su(n)$. Thus, the elements of $u_n(1)$ are uniquely determined by the θ-independent $\epsilon^0(x)$, which themselves form, as conjugacy classes, the local Lie algebra isomorphic to the usual commutative $u(1)$-gauge algebra. Analogous identifications are valid for other cases in (A.3) too.

The local gauge groups $U_\star(n)$, NC$SU(n)$, $U^n_\star(1)$ and NC$SU_n(1)$ are defined by taking the star-exponent of the corresponding ideal:

$$\begin{aligned}U_\star(n)&=\{\exp[\mathrm{i}\epsilon(x)],\,\epsilon(x)\in u_\star(n)\}\ ,\\ \mathrm{NC}SU(n)&=\{\exp[\mathrm{i}\epsilon(x)],\,\epsilon(x)\in \mathrm{NC}su(n)\}\ ,\end{aligned} \tag{A.4}$$

$$\begin{aligned}U^n_\star(1)&=\{\exp[\mathrm{i}\epsilon(x)],\,\epsilon(x)\in u^n_\star(1)\}\ ,\\ \mathrm{NC}SU_n(1)&=\{\exp[\mathrm{i}\epsilon(x)],\,\epsilon(x)\in \mathbf{1}^n_\star]\}\ .\end{aligned} \tag{A.5}$$

The corresponding factor-groups

$$U_n(1)\ =\ U_\star(n)/\mathrm{NC}SU(n)\ ,\qquad SU(n)\ =\ U_\star(n)/U^n_\star(1) \tag{A.6}$$

are all isomorphic to the usual gauge groups as is indicated by the notation. Thus, we can write

$$U_\star(n)\ =\ U_n(1)\star\mathrm{NC}SU(n)\ =\ SU(n)\star U^n_\star(1)\ . \tag{A.7}$$

The meaning of the first equality is the following: any element $U(x)\in U_\star(n)$ can be written (non-uniquely) as a product $U(x)=U'(x)\star V(x)$, where $U'(x)$ is some representant of a given conjugacy class and some element $V(x)$ from NC$SU(n)$. Alternatively, we can consider $U_\star(n)$ as a NC$SU(n)$-principal bundle over the set $U(1)$ of conjugacy classes: NC$SU(n)\to U_\star(n)\to U_n(1)$. More explicitly any element of $U_*(n)$ can be uniquely written as

$$\begin{aligned}U(x,\theta)&=\mathrm{e}_\star^{\mathrm{i}\epsilon_0(x)\mathbf{1}_n+\mathrm{i}\epsilon_1(x,\theta)\mathbf{1}_n+\mathrm{i}\epsilon_a(x,\theta)T^a}\\ &=\mathrm{e}^{\mathrm{i}\epsilon_0(x)\mathbf{1}_n}\star\mathrm{e}_\star^{\mathrm{i}\tilde\epsilon^1(x,\theta)\mathbf{1}_n+\mathrm{i}\tilde\epsilon_a(x,\theta)T^a}\ .\end{aligned} \tag{A.8}$$

Here $\mathrm{e}^{\mathrm{i}\epsilon(x)}$ denotes the usual exponent, whereas $\mathrm{e}_\star^{\mathrm{i}\epsilon(x,\theta)}$ is the $\star$-exponent. The second factor on the right-hand side of the second line of (A.8) is an element of the invariant subgroup NC$SU(n)$ being the $\star$-exponent of elements in NC$su(n)$. It follows that the elements of the factor-group $U_\star(n)/SU_\star(n)$ are uniquely specified by the first factor on the right-hand side. The product of two elements (A.8) is

$$\begin{aligned}&\left(\mathrm{e}^{\mathrm{i}\alpha_0(x)\mathbf{1}_n}\star\mathrm{e}_\star^{\mathrm{i}\tilde\alpha^1_0(x,\theta)\mathbf{1}_n+\mathrm{i}\tilde\alpha_a(x,\theta)T^a}\right)\\ &\star\left(\mathrm{e}^{\mathrm{i}\beta_0(x)\mathbf{1}_n}\star\mathrm{e}_\star^{\mathrm{i}\tilde\beta^1_0(x,\theta)\mathbf{1}_n+\mathrm{i}\tilde\alpha_a(x,\theta)T^a}\right)\\ &=\mathrm{e}^{\mathrm{i}(\alpha_0(x)+\beta_0(x))\mathbf{1}_n}\star\mathrm{e}_\star^{\mathrm{i}\tilde\gamma^1_0(x,\theta)\mathbf{1}_n+\mathrm{i}\tilde\gamma^0_c(x,\theta)T^c}\ ,\end{aligned} \tag{A.9}$$

where $\tilde\gamma^1_0(x,\theta)$ and $\tilde\gamma_c(x,\theta)$ depend on all other functions appearing in the left-hand side, $\tilde\alpha_A(x,\theta)$, $\tilde\beta_A(x,\theta)$ as well as $\alpha_0(x)$ and $\beta_0(x)$. However, the $U(1)$ factors specifying the elements of $U_\star(n)/SU_\star(n)$ on left-hand side only depend on $\alpha_0(x)$ and $\beta_0(x)$ and the $U_\star(n)/SU_\star(n)$ element on the right-hand side is determined by $\mathrm{e}^{\mathrm{i}(\alpha_0(x)+\beta_0(x))\mathbf{1}_n}$. We see that the factor-group is *isomorphic* to the usual commutative local gauge group: $U_\star(n)/\mathrm{NC}SU(n)=U_n(1)$. We stress that in our NCSM construction we have only used the $U_n(1)$ and NC$SU(n)$ sub-groups.

Realization of the $U(1)$ gauge symmetry

The formulas (A.8) and (A.9) induce the *one-dimensional* representation π of the $U_\star(n)$ group:

$$\pi(U(x,\theta))\ =\ \pi(\mathrm{e}_\star^{\mathrm{i}\alpha_0(x)\mathbf{1}_n+\mathrm{i}\alpha_1(x,\theta)\mathbf{1}_n+\mathrm{i}\alpha_a(x,\theta)T^a})\ =\ \alpha_0(x)\ , \tag{A.10}$$

possessing the property

$$\pi(\mathrm{e}_\star^{\mathrm{i}\alpha_0(x)\mathbf{1}_n+\dots}\star\mathrm{e}_\star^{\mathrm{i}\beta_0(x)\mathbf{1}_n+\dots})=\alpha_0(x)\ +\ \beta_0(x)\ . \tag{A.11}$$

This representation is realized on the gauge potentials

$$\begin{aligned}A(x,\theta)&\equiv A_A(x,\theta)T^A\\ &=A_0(x)\mathbf{1}_n+A_1(x,\theta)\mathbf{1}_n+\mathrm{i}A^0_a(x,\theta)T^a,\end{aligned}$$

which under $U_\star(n)$ transforms in the usual way:

$$\begin{aligned}A(x,\theta)\to\ &U(x,\theta)\star A(x,\theta)\star U^{-1}(x,\theta)\\ &+U(x,\theta)\star\mathrm{d}U^{-1}(x,\theta)\ .\end{aligned} \tag{A.12}$$

It can be seen that under (A.12) the θ-independent part of the gauge field $A_0(x)$ transforms as a usual $U(1)$ gauge field:

$$A_0(x)\ \to\ A_0(x)\ +\ \mathrm{d}\alpha_0(x)\ . \tag{A.13}$$

Then we can require that the θ-independent complex scalar Higgsac field $\Phi(x)$ under (A.12) transform as

$$\Phi\ \to\ e^{\mathrm{i}q\alpha_0(x)}\,\Phi(x)\ ,\ q\ -\ \text{constant}\ . \tag{A.14}$$

We stress that there are no $\star$-products on the right-hand side! An autonomous $U_n(1)$ gauge subsystem can be desribed by the Higgsac action

$$S[A_0,\Phi]=\int\mathrm{d}x\,[(D(A_0)\Phi(x))^\dagger(D(A_0)\Phi(x))-V(\Phi^\dagger\Phi)]\ , \tag{A.15}$$

where $V(.,.)$ is a convenient Higgs potential and $D(A_0)=\mathrm{d}+\mathrm{i}qA_0(x)$ is the θ-independent $U_n(1)$ part of the corresponding covariant derivative affiliated to the (full) gauge potential $A(x,\theta)$.

Based on the above discussions, the following notes are in order.

(1) As desired, the Higgsac action $S[A_0,\Phi]$ is $U_\star(n)$-invariant. However, the Higgsac field may interact with other matter fields *only* indirectly, via the θ-independent part $A_0(x)$ of the corresponding gauge field.
(2) The charge q of the Higgsac field $\Phi(x)$ is unspecified. Moreover, the Higgsac field may interact with more (θ-independent parts of) gauge fields with unspecified charges.
(3) The charges are determined by constant gauge transformations which are a part of the θ-independent factor of the gauge symmetry. The θ-dependent $U_n(1)$ fields $A^1_\theta(x,\theta)$ do not feel *any* $U(1)$ charge.
(4) The construction described above can be repeated for any non-commutative associative algebra of functions possessing a suitable filtration.

B Symmetry reduction

Let us consider a system with non-commutative gauge symmetry given as the direct product of two gauge groups:

$$\begin{aligned}&U_\star(n)\times U_\star(m) \\ &\ = (U_n(1)\times U_m(1))\star(\mathrm{NC}SU(n)\times \mathrm{NC}SU(m))\,,\end{aligned} \tag{B.1}$$

where the subscripts of the $U(1)$ factors indicate to which $U_\star(\cdot)$ gauge group they belong.

The symmetry reduction consists of replacing the two independent $U(1)$ factors

$$\begin{aligned}U_n(1) &= \left\{\exp\left[\frac{\mathrm{i}}{2}\epsilon^0_n(x)\mathbf{1}_n\right]\right\}\ ,\ \epsilon^0_n(x)\in\mathcal{A}\ ,\\ U_m(1) &= \left\{\exp\left[\frac{\mathrm{i}}{2}\epsilon^0_m(x)\mathbf{1}_m\right]\right\}\ ,\ \epsilon^0_m(x)\in\mathcal{A}\ ,\end{aligned} \tag{B.2}$$

by a "diagonal" one specified by putting $\epsilon^0_n(x)=\frac{1}{n}\epsilon^0(x)$ and $\epsilon^0_m(x)=\frac{1}{m}\epsilon^0(x)$ with $\epsilon^0(x)\in\mathcal{A}$:

$$\begin{aligned}&(U_n(1)\times U_m(1))_d \\ &\equiv\ \left\{\exp\left[\frac{\mathrm{i}}{2}\left(\frac{1}{n}\mathbf{1}_n\oplus\frac{1}{m}\mathbf{1}_m\right)\epsilon^0(x)\right]\right\} \\ &=\left\{\exp\left[\frac{\mathrm{i}}{2n}\epsilon^0(x)\mathbf{1}_n\right]\right\}\times\left\{\exp\frac{\mathrm{i}}{2m}\epsilon^0(x)\mathbf{1}_m]\right\}\ ,\end{aligned} \tag{B.3}$$

where $\epsilon^0(x)\in\mathcal{A}$ and the symbol $\oplus$ denotes the direct sum. In $U_n(1)$ we can introduce the determinant by

$$\det\left(\exp\left[\frac{\mathrm{i}}{2}\epsilon^0_n(x)\mathbf{1}_n\right]\right)=\exp\left[\frac{\mathrm{i}}{2}n\epsilon^0_n(x)\right].$$

The $\frac{1}{n}$ factors guarantee that

$$\begin{aligned}\det\left(\exp\left[\frac{\mathrm{i}}{2n}\epsilon^0(x)\mathbf{1}_n\right]\right)&=\det\left(\exp\left[\frac{\mathrm{i}}{2m}\epsilon^0(x)\mathbf{1}_m\right]\right)\\ &=\exp\left[\frac{\mathrm{i}}{2}\epsilon^0(x)\right]\end{aligned}$$

is a representation of $(U_n(1)\times U_m(1))_d$. After the symmetry reduction we are left with the gauge group

$$\begin{aligned}&(U_\star(n)\times U_\star(m))_d \\ &\equiv(U_n(1)\times U_m(1))_d\star(\mathrm{NC}SU(n)\times\mathrm{NC}SU(m)) \\ &=(U_\star(n)\times U_\star(m))/(\det(U_n(1))_d=\det(U_m(1))_d\ .\end{aligned} \tag{B.4}$$

In other words, the gauge groups $\mathrm{NC}SU(n)$ and $\mathrm{NC}SU(m)$ are not supplemented by two independent factors $U_n(1)$ and $U_m(1)$ but only by one diagonal factor $(U_n(1)\times U_m(1))_d$ containing strictly related factors $(U_n(1))_d$ and $(U_m(1))_d$ with equal determinants.

In the language of gauge fields this means the following. Originally, we have two gauge fields $A^n_\mu(x)$ and $A^m_\mu(x)$ sharing the gauge transformations of $U_n(1)=U_\star(n)/\mathrm{NC}SU(n)$ and $U_m(1)=U_\star(m)/\mathrm{NC}SU(m)$: they can be identified with the θ-independent parts of the $\mathbf{1}_n$ and $\mathbf{1}_m$ components of the $U_\star(n)$ and $U_\star(m)$ gauge fields, respectively. Under the $U_n(1)$ and $U_m(1)$ gauge transformations they transform as

$$\begin{aligned}A^n_\mu(x)&\to A^n_\mu(x)+\frac{\mathrm{i}}{2}g_n^{-1}\partial_\mu\epsilon^0_n(x)\mathbf{1}_n\ ,\ \epsilon^0_n(x)\in\mathcal{A}\ ,\\ A^m_\mu(x)&\to A^m_\mu(x)+\frac{\mathrm{i}}{2}g_m^{-1}\partial_\mu\epsilon^0_m(x)\mathbf{1}_m\ ,\ \epsilon^0_m(x)\in\mathcal{A}\ .\end{aligned} \tag{B.5}$$

After the symmetry reduction (B.2) and (B.3) there is one θ-independent gauge field $A^d_\mu(x)$ sharing the $(U_n(1)\times U_m(1))_d$ gauge symmetry:

$$\begin{aligned}A^d_\mu(x)&\equiv(A^n_\mu(x)\oplus A^m_\mu(x))_d\\ &=\left(\frac{g}{ng_n}\mathbf{1}_n\oplus\frac{g}{mg_m}\mathbf{1}_m\right)A_\mu(x)\ .\end{aligned} \tag{B.6}$$

The θ-independent field $A_\mu(x)$ transforms under $(U_n(1)\times U_m(1))_d$ gauge transformations generated by $\epsilon^0(x)$ as follows:

$$A_\mu(x)\ \to\ A_\mu(x)+\frac{\mathrm{i}}{2}g^{-1}\partial_\mu\epsilon^0(x)\ . \tag{B.7}$$

It is important that (B.5) coincides on $(U_n(1)\times U_m(1))_d$ transformations with (B.6) and (B.7). The constant g is specified by the equation

$$\frac{1}{g^2}\ =\ \frac{1}{n^2g_n^2}\ +\frac{1}{m^2g_m^2} \tag{B.8}$$

(this guarantees the proper normalization of the A_μ-field term in the Lagrangian).

C Symmetry reduction, fermionic part

In NCSM the symmetry reduction is mediated by a θ-independent Higgsac field $\Phi(x)\in\mathcal{A}$ possessing $(U_n(1)\times U_m(1))$ transformations:

$$\begin{aligned}&\Phi(x)\ \to\ u(x)\Phi(x)v^{-1}(x)\ =\ \Phi(x)\ ,\\ &u(x)=\det\exp\left[\frac{\mathrm{i}}{2}\epsilon^0_n(x)\mathbf{1}_n\right]\ ,\\ &v(x)=\det\exp\left[\frac{\mathrm{i}}{2}\epsilon^0_m(x)\mathbf{1}_m\right]\ .\end{aligned}$$

Note that there is no $\star$-product involved in the gauge transformation of the Higgsac field. However, for the $(U_n(1) \times U_m(1))_d$ transformations $\Phi(x) \to \Phi(x)$ (since $\epsilon_n^0(x) = \frac{1}{n}\epsilon^0(x)$, $\epsilon_m^0(x) = \frac{1}{m}\epsilon^0(x)$, and both phase factors cancel). Thus, the Higgsac field is *neutral* with respect to the residual gauge field $A_\mu(x)$. This is consistent with the observation that the $\Phi(x)$ field covariant derivative $(\partial_\mu + \mathrm{i}g_n \det A_\mu^n(x) - \mathrm{i}g_m \det A_\mu^m(x))$ does not transform under $(U_n(1) \times U_m(1))_d$ gauge transformations.

Let us now consider the matter fields $\Psi_u(x)$ and $\Psi_d(x)$ transforming under a $U_\star(n) \times U_\star(m)$ gauge transformation as follows:

$$\begin{aligned} \Psi_u(x) &\to U(x)\Psi_u(x)V^{-1}(x)\ , \\ \Psi_d(x) &\to \Psi_d(x)V^{-1}(x)\ , \end{aligned} \tag{C.1}$$

with $U(x) \in U_\star(n)$ and $V(x) \in U_\star(m)$. Their NC$SU(n) \times$ NC$SU(m)$ orbits are

$$\Big\{U(x)\Psi_u(x)V^{-1}(x),\quad U(x) \in \mathrm{NC}SU(n), \qquad V(x) \in \mathrm{NC}SU(m)\Big\}\ , \tag{C.2}$$
$$\{\Psi_d(x)V^{-1},\ V(x) \in \mathrm{NC}SU(m)\}\ .$$

This means that if $\psi_u(x)$ and $\psi_d(x)$ are representatives of the classes in question, then

$$\Psi_u(x) = U(x)\psi_u(x)V^{-1}(x)\ ,\ \Psi_d(x) = \psi_d(x)V^{-1}(x)\ , \tag{C.3}$$

with $U(x) \in \mathrm{NC}SU(n)$ and $V(x) \in \mathrm{NC}SU(m)$. The fields $\psi_u(x)$ and $\psi_d(x)$ transform under the $U_n(1) \times U_m(1)$ gauge transformation (B.5) as follows:

$$\begin{aligned} \psi_u(x) &\to \exp\left[\frac{\mathrm{i}}{2}\epsilon_n^0(x)\mathbf{1}_n\right]\psi_u(x)\exp\left[-\frac{\mathrm{i}}{2}\epsilon_m^0(x)\mathbf{1}_m\right]\ , \\ \psi_d(x) &\to \psi_d(x)\exp\left[-\frac{\mathrm{i}}{2}\epsilon_m^0(x)\mathbf{1}_m\right]\ . \end{aligned} \tag{C.4}$$

Restricting (C.4) to $(U_n(1) \times U_m(1))_d$ transformations by putting $\epsilon_n^0(x) = \frac{1}{n}\epsilon^0(x)$ and $\epsilon_m^0(x) = \frac{1}{m}\epsilon^0(x)$ we find that the orbits transform as

$$\begin{aligned} \psi_u(x) &\to \exp\left[\frac{\mathrm{i}}{2n}\epsilon^0(x)\mathbf{1}_n\right]\ \psi_u(x)\ \exp\left[-\frac{\mathrm{i}}{2m}\epsilon^0(x)\mathbf{1}_m\right]\ , \\ \psi_d(x) &\to \psi_d(x)\ \exp\left[-\frac{\mathrm{i}}{2m}\epsilon^0(x)\mathbf{1}_m\right]\ . \end{aligned} \tag{C.5}$$

Comparing this with (B.7) we see that they possess fractional A_μ-field charges:

$$q_u = \frac{g}{n} - \frac{g}{m}\ ,\ q_d = -\frac{g}{m}\ . \tag{C.6}$$

This is the solution of the *fractional charge* mystery in NC QFT: they appear as charges of the θ-independent residual gauge field $A_\mu(x)$ which transforms like a commutative $U(1)$ gauge field and can interact with the fields $\psi_u(x)$ and $\psi_d(x)$ possessing fractional charges. We can extend this as follows. The fields $\Psi_u(x)$ and $\Psi_d(x)$ themselves, and not only $\psi_u(x)$ and $\psi_d(x)$, possess the fractional charges q_u and q_d given above. This is reasonable, since charges are determined by global transformations with $\epsilon^0(x) = \text{const}$, and (C.5) and (C.6) with constant $\epsilon^0(x)$ are valid directly for the matter fields $\Psi_u(x)$ and $\Psi_d(x)$. The exact values of charges given in (C.3) can be read directly from the field transformation law (C.3). This simple rule is valid for any field.

We note that these are exactly the formulas discussed in Sects. 3 and 4. For example, the formula for the charge of $\Psi_u(x)$ given there ($q_u = \frac{1}{2}(g_n \cos\delta_{nm} - g_m \sin\delta_{nm})$, $\tan\delta_{nm} = ng_n/mg_m$) is identical to (C.6). However, the motivation presented here is "kinematical"; being based only on symmetry considerations, the symmetry-reducing part of the Lagrangian is not specified. We see that the symmetry reduction is related only to the θ-independent parts of fields sharing the corresponding commutative $U_n(1) \times U_m(1)$ factor-group symmetries.

References

1. M. Chaichian, P. Prešnajder, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Lett. B **526**, 132 (2002), hep-th/0107037
2. M. Hayakawa, Perturbative Analysis on Infra-red and Ultraviolet Aspects of Non-commutative QED on $\mathbb{R}^4$, hep-th/9912167; Phys. Lett. B **478**, 394 (2000), hep-th/9912094
3. M.M. Sheikh-Jabbari, Phys. Rev. Lett. **84**, 5265 (2000), hep-th/0001167
4. T. Filk, Phys. Lett. B **376**, 53 (1996)
5. M. Chaichian, A. Demichev, P. Prešnajder, Nucl. Phys. B **567**, 360 (2000), hep-th/9812180
6. C.P. Martin, D. Sanchez-Ruiz, Phys. Rev. Lett. **83**, 476 (1999), hep-th/9903077; C.P. Martin, F. Ruiz Ruiz, Nucl. Phys. B **597**, 197 (2001), hep-th/0007131
7. S. Minwalla, M. Van Raamsdonk, N. Seiberg, JHEP **0002**, 020 (2000), hep-th/9912072
8. M. Van Raamsdonk, N. Seiberg, JHEP **0003**, 035 (2000), hep-th/0002186
9. I.Ya. Aref'eva, D.M. Belov, A.S. Koshelev, Phys. Lett. B **476**, 431 (2000), hep-th/9912075; A. Micu, M.M. Sheikh-Jabbari, JHEP **01**, 025 (2001), hep-th/0008057
10. I. Chepelev, R. Roiban, JHEP **0005**, 037 (2000), hep-th/9911098; **0103**, 001 (2001), hep-th/0008090
11. I.Ya. Aref'eva, D.M. Belov, A.S. Koshelev, A Note on UV/IR for Non-commutative Complex Scalar Field, hep-th/0001215
12. Ihab.F. Riad, M.M. Sheikh-Jabbari, JHEP **0008**, 045 (2000), hep-th/0008132
13. A. Armoni, Nucl. Phys. B **593**, 229 (2001), hep-th/0005208
14. L. Bonora, M. Salizzoni, Phys. Lett. B **504**, 80 (2001), hep-th/0011088
15. N. Seiberg, E. Witten, JHEP **9909**, 032 (1999), hep-th/9908142
16. N. Seiberg, L. Susskind, N. Toumbas, JHEP **0006**, 044 (2000), hep-th/0005015; J. Gomis, T. Mehen, Nucl. Phys. B **591**, 265 (2000), hep-th/0005129; M. Chaichian, A. Demichev, P. Prešnajder, A. Tureanu, Eur. Phys. J. C **20**, 767 (2001), hep-th/0007156
17. L. Griguolo, M. Pietroni, Hard Non-commutative Loops Resummation, hep-th/0102070; JHEP **0105**, 032 (2001), hep-th/0104217

18. A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda, R. Wulkenhaar, JHEP **0106**, 013 (2001), hep-th/0104097
19. I. Mocioiu, M. Pospelov, R. Roiban, Phys. Lett. B **489**, 390 (2000), hep-ph/0005191
20. N. Chair, M.M. Sheikh-Jabbari, Phys. Lett. B **504**, 141 (2001), hep-th/0009037
21. M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001), hep-th/0010175
22. J.L. Hewett, F.J. Petriello, T.G. Rizzo, Signals for Non-commutative Interactions at Linear Colliders, hep-ph/0010354
23. P. Mathews, Phys. Rev. D **63**, 075007 (2001), hep-ph/0011332
24. M. Chaichian, A. Demichev, P. Prešnajder, M.M. Sheikh-Jabbari, A. Tureanu, Nucl. Phys. B **611**, 383 (2001), hep-th/0101209
25. S.M. Carroll, J.A. Harvey, V. Alan Kostelecky, C.D. Lane, T. Okamoto, Phys. Rev. Lett. **87**, 141601 (2001), hep-th/0105082
26. H. Grosse, Y. Liao, Phys. Lett. B **520**, 63 (2001), hep-ph/0104260; Phys. Rev. D **64**, 115007 (2001), hep-ph/0105090
27. A. Anisimov, T. Banks, M. Dine, M. Graesse, Phys. Rev. D **65**, 085032 (2002), hep-ph/0106356
28. A. Connes, J. Lott, Nucl. Phys. Proc. Supl. B **18**, 29 (1991); A. Chamsedine, A. Connes, Phys. Rev. Lett. **77**, 4868 (1996); A. Alvarez, J.M. Gracia-Bondia, C.P. Martin, Phys. Lett. B **364**, 33 (1995); C.P. Martin, J.M. Gracia-Bondia, J.C. Varilly, Nucl. Phys. Proc. Suppl. B **18**, 29 (1991)
29. S. Lazzarini, T. Schücker, Phys. Lett. B **510**, 277 (2001), hep-th/0104038
30. S. Terashima, Phys. Lett. B **482**, 276 (2000), hep-th/0002119
31. M.M. Sheikh-Jabbari, JHEP **9906**, 015 (1999), hep-th/9903107
32. M. Fukugita, S. Yazaki, Phys. Rev. D **36**, 3817 (1987)
33. G. Altarelli, R. Barbieri, F. Caravaglios, Int. J. Mod. Phys. A **13**, 1031 (1998), hep-ph/9712368
34. G. Altarelli, F. Caravaglios, G.F. Giudice, P. Gambino, G. Ridolfi, JHEP **0106**, 018 (2001), hep-ph/0106029
35. D.E. Groom et al., Eur. Phys. J. C **15**, 1 (2000)
36. F. Ardalan, N. Sadooghi, Int. J. Mod. Phys. A **16**, 3151 (2001), hep-th/0002143; A **17**, 123 (2002), hep-th/0009233; J.M. Gracia-Bondia, C.P. Martin, Phys. Lett. B **479**, 321 (2000), hep-th/0002171. L. Bonora, M. Schnabl, A. Tomasiello, Phys. Lett. B **485**, 311 (2000), hep-th/0002210
37. K. Intriligator, J. Kumar, Nucl. Phys. B **620**, 315 (2002), hep-th/0107199
38. X. Calmet, B. Jurco, P. Schupp, J. Wess, M. Wohlgenannt, Eur. Phys. J. C **23**, 363 (2002), hep-th/0111115
39. P. Aschieri, B. Jurco, P. Schupp, J. Wess, Non-Commutative GUTs, Standard Model and C,P,T, hep-th/0205214; C.P. Martin, The Gauge Anomaly and the Seiberg–Witten Map, hep-th/0211164; P. Schupp, J. Trampetic, J. Wess, G. Raffelt, The photon–neutrino interaction in non-commutative gauge field theory and astrophysical bounds, hep-ph/0212292